Elliptic deformations of quantum Virasoro and W_n algebras

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Extension of work by J.A, L.F., M. Rossi, P. Sorba, 1997-99

References:

Deformed Virasoro algebras from elliptic quantum algebras
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arXiv: 1607-05050

Plan :

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I -1 INTRODUCTION: THE FIRST DEFORMED VIRASORO ALGEBRA

Reference : J. Shiraishi, H.Kubo, H. Awata, S. Odake ; Lett. Math. Phys. **38** (1996), 33

Notion discussed in e.g. Curtright-Zachos 1990. More precisely derived from construction of Virasoro Poisson algebra on extended center of affine A1 algebra at critical center c=-2

Extended center of affine q-deformed A_1 algebra again at c=-2 : Reshetikhin Semenov-Tjan-Shanskii 1990 ; Poisson structure : Frenkel Reshetikhin 1996 ; Quantized by SKAO and Feigin-Frenkel 1996 ;
Extension to q-W_n algebras by same authors and Awata-Kubo-Odake-Shiraishi 1996 : extended center at c=-n for affine q-deformed sl(n) algebra.

Proposed structure :

Algebra generated by the generating functional T(z) satisfying the relation

$$f(z/w)T(z)T(w) - T(w)T(z)f(z/w) = -\frac{(1-q)(1-t^{-1})}{1-p}\left[\delta\left(\frac{pw}{z}\right) - \delta\left(\frac{w}{pz}\right)\right]$$

Structure function is given by

$$f(x) = \exp\left(\sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})}{1+p^n}\frac{x^n}{n}\right)$$

t=p/q

Warning : notations here are « old reckoning » : p,q (AKOS) = q^2, p^{-1} (our soon-to-come parameters)

Classical limit *ln p = lnq* ; Virasoro limit *p,q → 1+ o(ε)*, *ln p/ln q =b ; T(z) = 2 + ε^2 (t(z)+...)*

Occurs as e.g.

Symmetries for restricted SOS models (Lukyanov 1996) ;
Natural operators acting on eigenvectors for Ruijsenaar Schneider models ; hence connection with Macdonald and Koornwinder polynomials (SKAO 1996) ;
Natural algebraic structure for partner of 5D gauge field theory in extension of AGT conjecture (Awata-Yamada 2010, Nieri 2015). Hence connection also to q-Painlevé : See new work : Bershtein-Shchechkin 1608.02566

Quantization by construction of vertex operators using current algebra construction of Virasoro/Wn and q-deforming it (SKAO, FF).

Alternative : Direct embedding into larger algebraic structure ? Started in previous papers AFRS 97-99.

General idea : SKAO formula has 2 parameters p,q and constituent blocks of elliptic functions :
Ratio of structure functions f(x)/f(x^{-1}) is ratio of elliptic Jacobi Theta functions.

Hence suggests quantization of classical DVA naturally inserted into *elliptic affine algebra* instead of quantum affine algebra.

Consider elliptic gl(N) algebra with generic N. A priori leads to deformation of W. But restrict here to spin 1 generator, or N=2. Extension to full W_N in project. Partially realized in AFRS 97-99. Notion of quantum « powers » to be refined.

I -2 THE ELLIPTIC ALGEBRA $A_{qp}(gl(N)_c$

Original proposition for gl(2) by Foda-Iohara-Jimbo-Kedem-Miwa-Yan 1994. Goes to sl(2) by factoring out q-determinant. To be commented later on.

Extended to gl(N)
by Jimbo-Kono-Odake-Shiraishi 1999. Justification of quasi Hopf structure in Arnaudon-Buffenoir-Ragoucy-Roche 1998 = identification of Drinfel'd twist. sl(N) ??? q-determinant ???

Lax matrix encapsulates generators as :

$$L(z) = \begin{pmatrix} L_{11}(z) & \cdots & L_{1N}(z) \\ \vdots & & \vdots \\ L_{N1}(z) & \cdots & L_{NN}(z) \end{pmatrix} .$$

Exchange relation

$$\widehat{R}_{12}(z/w)\, L_1(z)\, L_2(w) = L_2(w)\, L_1(z)\, \widehat{R}^*_{12}(z/w) \,,$$

R-matrix (Baxter 1981, Chudnovskii-Chudnovskii 1981) , in term of Jacobi theta functions with rational characteristics. g,h related with periods of elliptic functions ;

$$\mathcal{Z}(z,q,p) = z^{2/N-2}\frac{1}{\kappa(z^2)}\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\zeta,\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\xi+\zeta,\tau)} \sum_{(\alpha_1,\alpha_2)\in\mathbb{Z}_N\times\mathbb{Z}_N} W_{(\alpha_1,\alpha_2)}(\xi,\zeta,\tau)\; I_{(\alpha_1,\alpha_2)}\otimes I^{-1}_{(\alpha_1,\alpha_2)}$$

$$z = e^{i\pi\xi}\,, \qquad q = e^{i\pi\zeta}\,, \qquad p = e^{2i\pi\tau}\,.$$

$$W_{(\alpha_1,\alpha_2)}(\xi,\zeta,\tau) = \frac{\vartheta\begin{bmatrix}\frac{1}{2}+\alpha_1/N\\ \frac{1}{2}+\alpha_2/N\end{bmatrix}(\xi+\zeta/N,\tau)}{N\vartheta\begin{bmatrix}\frac{1}{2}+\alpha_1/N\\ \frac{1}{2}+\alpha_2/N\end{bmatrix}(\zeta/N,\tau)}$$

$$R(z,q,p) = (g^{\frac{1}{2}}\otimes g^{\frac{1}{2}})\mathcal{Z}(z,q,p)(g^{-\frac{1}{2}}\otimes g^{-\frac{1}{2}})\,; \quad g_{ij} = \omega^i\delta_{ij}$$

$$\widehat{R}_{12}(z) \equiv \widehat{R}_{12}(z,q,p) = \tau_N(q^{\frac{1}{2}}z^{-1})\, R_{12}(z,q,p)\,, \quad \tau_N(z) = z^{\frac{2}{N}-2}\,\frac{\Theta_{q^{2N}}(qz^2)}{\Theta_{q^{2N}}(qz^{-2})}\,.$$

$$\widehat{R}^*_{12}(z,q,p) = \widehat{R}_{12}(z,q,p^* = pq^{-2c})$$

$\Theta_p(z) = (z;p)_\infty \, (pz^{-1};p)_\infty \, (p;p)_\infty$ Here arises *quasi-Hopf* structure (recall e.g. dynamical quantum algebra).

Warning : R is unitary solution to Yang-Baxter equation, but does not enter into definition of elliptic algebra from quasi-Hopf structure. (Drinfel'd - twisted R matrix is non-unitary one!
See Arnaudon-Buffenoir-Ragoucy-Roche 1998). Seems not relevant but …

Relation with (untwisted) quantum group structure obtained by redefining :

$$L^+(z) = L(q^{\frac{c}{2}} z)\,,$$
$$L^-(z) = (g^{\frac{1}{2}} h g^{\frac{1}{2}})\, L(-p^{\frac{1}{2}} z)\, (g^{\frac{1}{2}} h g^{\frac{1}{2}})^{-1}\,.$$

and reads :

$$\widehat{R}_{12}(z/w)\, L_1^{\pm}(z)\, L_2^{\pm}(w) = L_2^{\pm}(w)\, L_1^{\pm}(z)\, \widehat{R}^*_{12}(z/w)\,,$$
$$\widehat{R}_{12}(q^{\frac{c}{2}} z/w)\, L_1^{+}(z)\, L_2^{-}(w) = L_2^{-}(w)\, L_1^{+}(z)\, \widehat{R}^*_{12}(q^{-\frac{c}{2}} z/w)\,.$$

Use in this formulation of non-unitary R matrix instead of R *NOW* leads to different structure due to different normalization. (to be kept in mind).

I -3 : THE QUANTUM DETERMINANT FOR N=2

FOR N = 2 Elliptic R-matrix evaluated at -1/q degenerates to **I +P$_{12}$** (permutation operator).
Equal to antisymmetrizer . Hence possible to define q-determinant :

$$\begin{aligned} \text{q-det}\, L(z) &= L_{11}(q^{-1}z)L_{22}(z) - L_{21}(q^{-1}z)L_{12}(z) \\ &= L_{22}(q^{-1}z)L_{11}(z) - L_{12}(q^{-1}z)L_{21}(z) \\ &= L_{11}(z)L_{22}(q^{-1}z) - L_{12}(z)L_{21}(q^{-1}z) \\ &= L_{22}(z)L_{11}(q^{-1}z) - L_{21}(z)L_{12}(q^{-1}z) \end{aligned}$$

Lies in center of quantum elliptic algebra. Allows to define inverse $\widehat{L}(q^{-1}z)L(z) = \text{q-det}\, L(z)\, \mathbb{I}_2\,.$

and comatrix :

$$\widehat{L}(z) = \begin{pmatrix} L_{22}(z) & -L_{12}(z) \\ -L_{21}(z) & L_{11}(z) \end{pmatrix}$$

I -4 THE STRATEGY

First step : define quadratic functional of Lax matrix, and conditions on parameters p,q,c such that quadratic functional close exchange algebra : CLOSURE CONDITIONS

Second step : define second (analytical) set of conditions on p,q,c such that exchange algebra become abelian : ABELIANITY CONDITIONS.

Then expand around set of conditions to get Poisson structure. Closure algebra then automatically yields quantization of Poisson structure.

Does one get classical DVA Poisson ? Does it quantize to DVA ?

II-1 THE CLOSURE RELATIONS

General result :

Theorem 3.1 *In the three-dimensional parameter space generated by q, p, c, consider the two-dimensional surfaces $\mathscr{S}_{mn}$ defined by the relation $(-p^{\frac{1}{2}})^m(-p^{*\frac{1}{2}})^n = q^{-N}$ for any integers $m, n \in \mathbb{Z}$, $n \neq 0$. On any surface $\mathscr{S}_{mn}$, the generators*

$$t_{mn}(z) = \mathrm{tr}\left((g^{\frac{1}{2}}hg^{\frac{1}{2}})^{-m}L((-p^{*\frac{1}{2}})^n z)(g^{\frac{1}{2}}hg^{\frac{1}{2}})^{-n}L(z)^{-1}\right)$$

realize an exchange algebra with the generators $L(w)$ of $\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$:

$$t_{mn}(z)\,L(w) = \frac{\mathcal{F}_{-m}(z/w)}{\mathcal{F}^*_n(z/w)}\,L(w)\,t_{mn}(z)$$

where

$$\mathcal{F}_a(x) = \begin{cases} \prod_{k=0}^{a-1}\mathcal{U}((-p^{\frac{1}{2}})^k x) & \text{for } a > 0 \\ \prod_{k=1}^{|a|}\mathcal{U}((-p^{\frac{1}{2}})^{-k} x)^{-1} & \text{for } a < 0 \end{cases}$$

and $\mathcal{F}^*_a(x) = \mathcal{F}_a(x)\big|_{p\to p^*}$. One sets $\mathcal{F}_0(x) = 1$.

Remark 1 : the previous results (AFRS 97-99) correspond to the case n=-1, m = kN -1 , k integer, where the contribution of the automorphism M= $g^{1/2}hg^{1/2}$ essentially vanish due to $M^N = 1$, the balance is actually reabsorbed in the redefinition L → L_ .

Remark 2 : when n = -m = ± 1 closure relation yields c = ± N and leads directly to extended center, i.e. commutation of t_{nm}(w) with L(z). c=-N known in affine algebras. c=N not yet known ; seems specific to elliptic algebras.

Remark 3 : this extends to n=-m, n odd, c=N/n and $-p^{1/2} = q^{-Nk/n}$, k not equal to n-1, coprime with n with Bezout coefficients resp. b,b', bk+b'n = 1, and b+1 coprime with n : « weak extended center » (requires two conditions).

Implies commutation of generators t with themselves, seen later in abelianity section.

Remark 4 : Alternative closure relation constructed with automorphism M → g^aM, M' → g^bM', denoting M and M' as resp. left and right automorphism in definition of t_{nm} . Uses antisymmetry of R matrix. Adds extra $(–)^{a+b}$ sign to closure relation and phase $e^{i(a+b)\pi/N}$ to exchange algebra.

Remark 5 : Liouville formula at N=2

$t_{0,2}(z) = \frac{1}{2}\,\mathrm{tr}\left(L(q^{-2}z)L(z)^{-1}\right)$ lies in the center of the elliptic quantum algebra.

No closure condition required ! Related to the q-det through

$$t_{0,2}(z) = \frac{\text{q-det}\,L(q^{-1}z)}{\text{q-det}\,L(z)}\,.$$

Remark 6 : Alternative construction :

Theorem 3.3 *On the surface* $\mathscr{S}_{mn}$, *the generators*

$$t^*_{-n,-m}(z) = \mathrm{tr}\left((g^{\frac{1}{2}}hg^{\frac{1}{2}})^n L((-p^{\frac{1}{2}})^{-m}z)^{-1}(g^{\frac{1}{2}}hg^{\frac{1}{2}})^m L(z)\right)$$

realize an exchange algebra with the generators $L(w)$ *of* $\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$:

$$t^*_{-n,-m}(z)\, L(w) = \frac{\mathcal{F}^*_n(z/w)}{\mathcal{F}_{-m}(z/w)}\, L(w)\, t^*_{-n,-m}(z)$$

From which it is immediate to get :

Proposition 3.4 *On the surface* $\mathscr{S}_{mn}$, *the generators* $t_{mn}(z)t^*_{-n,-m}(z)$ *lie in the center of* $\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$.

In addition t*(z), t*(w) realize same exchange algebra as t(z), t(w) .

Connection between t and t* generators suggested by above results, seems to be $t^{-1} \sim t^*$.

However only possible to establish assuming closure relation and N=2 as :

$$t^*_{-n,-m}(z) = \frac{q\text{-}\det L(qz)}{q\text{-}\det L((-p^{\frac{1}{2}})^{-m}z)}\, t_{mn}(qz)$$

Inverse structure function in fact realized through modular relation U(z) ~ 1/U(qz). Quite peculiar to N=2 as degeneracy between N/2 and 2.

N>2 ?

Exchange relations for t(z), t(w) follow immediately :

Corollary 3.5 *On the surface $\mathscr{S}_{mn}$, the generators*

$$t_{mn}(z) = \mathrm{tr}\left((g^{\frac{1}{2}}hg^{\frac{1}{2}})^{-m}L((-p^{*\frac{1}{2}})^{n}z)(g^{\frac{1}{2}}hg^{\frac{1}{2}})^{-n}L(z)^{-1}\right)$$

close a quadratic subalgebra in $\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$:

$$t_{mn}(z)\,t_{mn}(w) = \mathcal{Y}_{mn}(z/w)\,t_{mn}(w)\,t_{mn}(z)$$

with

$$\mathcal{Y}_{mn}(x) = \frac{\mathcal{F}^*_n(x)\mathcal{F}^*_{-n}(x)}{\mathcal{F}_m(x)\mathcal{F}_{-m}(x)} = \frac{\displaystyle\prod_{k=1}^{|m|}\mathcal{U}((-p^{\frac{1}{2}})^{-k}x)\prod_{k'=0}^{|n|-1}\mathcal{U}((-p^{*\frac{1}{2}})^{k'}x)}{\displaystyle\prod_{k=0}^{|m|-1}\mathcal{U}((-p^{\frac{1}{2}})^{k}x)\prod_{k'=1}^{|n|}\mathcal{U}((-p^{*\frac{1}{2}})^{-k'}x)}$$

Important point : For N=2 an explicit factorization is available :

$$\mathcal{Y}_{mn}(x) = \frac{g_{mn}(x^2)}{g_{mn}(x^{-2})} \quad \textit{with} \quad g_{mn}(z) = g_{corr}(z)\Big(\prod_{k=1}^{|m|-1} g^{(k)}(z)\Big)^2\Big(\prod_{k=1}^{|n|-1} g^{*(k)}(z)\Big)^{-2}$$

where

$$g^{(k)}(z) = \exp\Big(\sum_{\ell=1}^{\infty}\frac{(1-p^{-k\ell})(1-(p^kq^2)^{\ell})}{1+q^{2\ell}}\,\frac{z^{\ell}}{\ell}\Big)$$
$$g^{*(k)}(z) = g^{(k)}(z)\Big|_{p\to p^*}$$
$$g_{corr}(z) = g^{(|m|)}(z)\left(g^{*(|n|)}(z)\right)^{-1}$$

Note the overall square !

Hence scaling limit can be defined :

Defining the scaling limit as $p = 1+\varepsilon$ and $q = 1+\eta\,\varepsilon$ with $\varepsilon \to 0$, we observe that

$$\begin{aligned} g^{(k)}(z) &= 1-\varepsilon^2\,\frac{k(k+2\eta)}{2}\frac{z}{(1-z)^2}+o(\varepsilon^2) \\ g^{*(k)}(z) &= 1-\varepsilon^2\,\frac{k(1-2\eta c)\big(k(1-2\eta c)+2\eta\big)}{2}\frac{z}{(1-z)^2}+o(\varepsilon^2) \end{aligned}$$

leading to

$$g_{mn}(z) = 1-\varepsilon^2\Big\{\beta_m-(1-2\eta c)^2\beta_n-2\eta^2c(1-2\eta c)n(n-2)\Big\}\frac{z}{(1-z)^2}+o(\varepsilon^2)$$

where

$$\beta_\ell = \frac{|\ell|(|\ell|-1)(2|\ell|-1)}{6}+\eta\ell(\ell-2).$$

Also true for N>2, with no full square expression for g_{mn} and more complicated factor for scaling limit. But scaling limit nevertheless gives exact structure function for quantum Virasoro algebra . Hence quadratic algebras characterized as DEFORMED VIRASORO ALGEBRAS. But DVA of Shiraishi Kubo Awata Odake elusive due to square ! And delicate issue with central extension to get exact scaling limit (BUT ! 1st term of g anyway fully controls centerless part of limit) .

Additional exchange relations on surface $\mathscr{S}_{mn}$

$$t_{mn}(z)\, t_{rs}(w) \;=\; \frac{\mathcal{F}^*_n(z/w)}{\mathcal{F}_{-m}(z/w)}\,\frac{\mathcal{F}_{-m}((-p^{*\frac{1}{2}})^s z/w)}{\mathcal{F}^*_n((-p^{*\frac{1}{2}})^s z/w)}\; t_{rs}(w)\, t_{mn}(z)$$

Yields coefficients Y if s=n, no requirement on r.

Abelianity relations now follow from examining possible cancellations of terms in above products. Essentially « vertical » cancellation between m-products, and separately n-products, except when m=± n where cross-cancellations may occur as already seen on example of extended centre and weak extended center.

Remark 6 : First immediate abelianity condition : for t_{nm} at n =0, |m| >1 and t^*_{nm} at n=0, |m| >1 . Not extended center : Only on-shell (closure relation obeyed). Requires only closure. Liouville formula = stronger statement (off-shell).

II-2 THE CRITICALITY RELATIONS

Particular case already seen m=-n, more developed here.

General case :

- *for* $|m|, |n| > 1$:

$$c = \frac{N}{nm}\big(\lambda' m - \lambda n\big)\ ,\quad -p^{\frac{1}{2}} = q^{-N\lambda/m}\ ,\quad -p^{*\frac{1}{2}} = q^{-N\lambda'/n} \tag{3.19}$$

where $\lambda, \lambda' \in \mathbb{Z} \setminus \{0\}$ *and* $\lambda + \lambda' = 1$.

- *for* $|n| = 1, |m| > 1$:

$$c = Nn\big(1 - \lambda(m+n)\big)\ ,\quad -p^{\frac{1}{2}} = q^{-N\lambda}\ ,\quad -p^{*\frac{1}{2}} = q^{-Nn(1-\lambda m)} \tag{3.20}$$

where $\lambda \in \mathbb{Z}/2$ *or* $\lambda \in \mathbb{Z}/u$, u *being any divisor of* m *or* $m+n$.

- *for* $|m| = 1, |n| > 1$:

$$c = Nm\big(\lambda'(n+m) - 1\big)\ ,\quad -p^{\frac{1}{2}} = q^{-Nm(1-\lambda' n)}\ ,\quad -p^{*\frac{1}{2}} = q^{-N\lambda'} \tag{3.21}$$

where $\lambda' \in \mathbb{Z}/2$ *or* $\lambda' \in \mathbb{Z}/u'$, u' *being any divisor of* n *or* $n+m$.

- *for* $m = n = \pm 1$, *formulas* (3.20)–(3.21) *also hold with* $\lambda, \lambda' \in \mathbb{Z}/2$ *and* $\lambda + \lambda' = 1$.

- *for* $m + n = 0$ *with* $n > 0$ *and odd:*

$$c = \frac{N}{n}\ ,\quad -p^{\frac{1}{2}} = q^{-\frac{n-1}{2n}N}\ ,\quad -p^{*\frac{1}{2}} = q^{-\frac{n+1}{2n}N}. \tag{3.22}$$

Remark 7 : Similar result holds for generators in Remark 4 up to overall sign. Same result holds for t* generators since same exchange algebra.

II-3 THE POISSON STRUCTURES

Obtained in usual way. Fix closure condition $\mathscr{S}_{mn}$. Then set quasi-abelianity relation as :
$p^{1-\epsilon} = q^{\alpha N\ell}$

and expand as

$$\big\{t(z), t(w)\big\}_\ell = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(t(z)t(w) - t(w)t(z)\big)$$ One gets :

$$\big\{t(z), t(w)\big\}_\ell = f_\ell(z/w)\, t(z)t(w) \qquad f_\ell(x) = 2N\ell(\ln q)\big(2I(x) - I(qx) - I(q^{-1}x) - (x \leftrightarrow x^{-1})\big)$$

$I(x)$ is given by the following expressions depending on the different cases of proposition

- *for* $|m|, |n| > 1$ $(\alpha = 2/m)$:

$$I(x) = \sum_{s=1}^{\infty}\left(w\,\frac{x^2q^{2Nsw/m}}{1 - x^2q^{2Nsw/m}} + w'\,\frac{x^2q^{2Nsw'/n}}{1 - x^2q^{2Nsw'/n}}\right) + \tfrac{1}{2}\,(w + w')\,\frac{x^2}{(1 - x^2)}$$

where $w = gcd(\ell, m)$ and $w' = gcd(\ell', n)$ (w and w' being taken positive).

- *for $|n| = 1, |m| > 1$ ($\alpha = 1$) and ℓ even:*

$$I(x) = \tfrac{1}{2}\, m(m+1) \left[\sum_{s=1}^{\infty} \frac{x^2 q^{2Ns}}{1 - x^2 q^{2Ns}} + \tfrac{1}{2} \frac{x^2}{1-x^2} \right] \tag{4.15}$$

while for ℓ odd

$$I(x) = \lfloor \tfrac{m}{2} \rfloor \left(\lfloor \tfrac{m}{2} \rfloor + 1 \right) \left(\sum_{s=1}^{\infty} \frac{x^2 q^{2Ns}}{1 - x^2 q^{2Ns}} + \tfrac{1}{2} \frac{x^2}{1-x^2} \right) + \lfloor \tfrac{m+1}{2} \rfloor^2 \sum_{s=0}^{\infty} \frac{x^2 q^{N(2s+1)}}{1 - x^2 q^{N(2s+1)}} \tag{4.16}$$

$\lfloor x \rfloor$ is the floor function (integer part) of x.

- *for $|n| = 1, |m| > 1$ ($\alpha = 2/u$ where u is any divisor of $k_{\rm sup}$ or $k_{\rm sup} + 1$ and g is defined as $k_{\rm sup}/u$ or $(k_{\rm sup} + 1)/u$ respectively):*

$$I(x) = g \left[\eta \sum_{s=1}^{\infty} \left(\frac{x^2 q^{2Ns}}{1 - x^2 q^{2Ns}} + gw \frac{x^2 q^{2Nsw/u}}{1 - x^2 q^{2Nsw/u}} \right) + \tfrac{1}{2} (gw + \eta) \frac{x^2}{1-x^2} \right] \tag{4.17}$$

where $\eta = 1$ when $gu = k_{\rm sup}$ and $\eta = -1$ when $gu = k_{\rm sup} + 1$, $w = gcd(\ell, u)$, see the definition of $k_{\rm sup}$ after formula (C.14).

- *for $|m| = 1, |n| > 1$ ($\alpha = 1$): the formulas are analogous to (4.15)-(4.16) (ℓ being replaced by ℓ' and m by n), up to a non relevant normalization factor that can be absorbed by a redefinition of ϵ.*

II-4 THE DEFORMED VIRASORO ALGEBRAS

Classical limit yields classical DVA in e.g. (4.15), (4.16), up to normalizations

Directly realize quantum DVA exchange algebra on closure surface ?

Not true, as mentioned earlier

only S^2 for m = -2 ! Worse for higher values of m.

$\mathcal{A}_{q,p,c}(\widehat{gl}_N)$: modified elliptic quantum algebra. Already defined above :

$$\widehat{R}_{12}(z/w)\, L_1^{\pm}(z)\, L_2^{\pm}(w) = L_2^{\pm}(w)\, L_1^{\pm}(z)\, \widehat{R}^*_{12}(z/w)\,,$$
$$\widehat{R}_{12}(q^{\frac{c}{2}} z/w)\, L_1^{+}(z)\, L_2^{-}(w) = L_2^{-}(w)\, L_1^{+}(z)\, \widehat{R}^*_{12}(q^{-\frac{c}{2}} z/w)\,.$$

this time with unitary R matrix. No relation now assumed between L^+ and L^-. Seen in Foda et al. (1995). as vertex operator construction of original elliptic algebra.

HENCE NOT EQUIVALENT TO $A_{qp}(gl(N))_c$. « vertex-operator algebra » from its construction.

Introduce now :

$$t_{mn}(z) = \mathrm{tr}\left((g^{\frac{1}{2}}hg^{\frac{1}{2}})^{-m+1}\mathcal{L}^+((-p^{\frac{1}{2}})^{n+1}q^{-nc-c/2}z)(g^{\frac{1}{2}}hg^{\frac{1}{2}})^{-n-1}\mathcal{L}^-(z)^{-1}\right)$$
$$t^*_{mn}(z) = \mathrm{tr}\left((g^{\frac{1}{2}}hg^{\frac{1}{2}})^{n-1}\mathcal{L}^-((-p^{\frac{1}{2}})^{-m-1}q^{\frac{c}{2}}z)^{-1}(g^{\frac{1}{2}}hg^{\frac{1}{2}})^{m+1}\mathcal{L}^+(z)\right)$$

Same closure conditions, conditions since exchange relations of generating matrices only differ by scalar factor. Abelianity conditions certainly modified.

For N=2, on closure surface n=-1, m=2: exchange algebra realizes exactly quantum DVA

Recall : p,q (SKAO) = q^2, p^{-1} (alternative elliptic algebra parameters, c(p,q) expressed by closure relation)

III OPEN ISSUES

RE : DVA

Interpretation of new algebra ; in particular Hopf structure ?

Connection with MacDonald polynomials ? Seen in vertex operator construction of SKAO DVA, how about other DVA's ? in particular those obtained from $A_{qp}(gl(N))_c$ with « squared » structure functions.

Central extensions ? Not directly gotten in abstract construction from algebra relations. Explicit realizations and/or resolution of cocycle condition (partially done in '98). Important e.g. for consistent linear scaling limit

RE : OTHER STRUCTURES

N>2 leads to possible constructions of q-W_{N-1} algebras. (proposed in e.g. Feigin-Frenkel, AKOS) Requires to define consistent higher-spin generators as (?) traces of multilinear objects. First attempts in '98.

Dynamical algebras $B_{qp\lambda}(gl(N))_c$: promising ?