Quantum Affine Algebras and $\ell$-root operators

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Quantum Affine Algebras: loop perspective

\[ U_q(\hat{\mathfrak{g}})/(c - 1) \cong U_q(\mathcal{L}\mathfrak{g}) \]

- \( \mathfrak{g} \): simple Lie algebra over \( \mathbb{C} \).
- \( i \in I \): nodes of Dynkin diagram of \( \mathfrak{g} \).
- \( q \in \mathbb{C}^\times \), transcendental.
- \( \mathcal{L}\mathfrak{g} = \mathfrak{g}[t, t^{-1}] \): loop algebra of \( \mathfrak{g} \).

Generators of quantum loop algebra \( U_q(\mathcal{L}\mathfrak{g}) \) in Drinfeld’s current presentation:

\[ x_i^-(z), \quad \phi_i^\pm(z), \quad x_i^+(z), \]

lowering “Cartan” raising

**Theorem [Beck]** There is a triangular decomposition

\[ U_q(\mathcal{L}\mathfrak{g}) \cong \mathbb{C} U^- \otimes U^0 \otimes U^+. \]

**Theorem [Chari-Pressley]** Every finite-dimensional \( U_q(\mathcal{L}\mathfrak{g}) \)-module is “highest weight” with respect to this triangular decomposition.
\( \ell \)-weights and \( q \)-characters

\[
x_i^-(z), \quad \phi_i^\pm(z), \quad x_i^+(z),
\]
lowering \quad "Cartan" \quad raising

\[
U_q(\mathcal{L}g) \cong_C \, U^- \otimes U^0 \otimes U^+, \quad \phi_i^\pm(z) = \sum_{r=0}^\infty \phi_i,\pm_r z^{\pm r}
\]
Eigenvalues of \( \phi_i^\pm(z) \) are called \( \ell \)-weights.

Let \( V \) be a \( U_q(\mathcal{L}g) \)-module of finite dimension.

Define the \( \ell \)-weight space \( V_\gamma \) by

\[
V_\gamma := \{ v \in V : \exists k \in \mathbb{N}, (\phi_i,\pm_m - \gamma_i,\pm_m)^k v = 0 \, \forall i \in I, m \geq 0 \}.
\]

Have \( \ell \)-weight decomposition and "q-character" [Frenkel & Reshetikhin]

\[
V = \bigoplus_\gamma V_\gamma, \quad \chi_q(V) := \sum_{\text{formal sum}} \gamma \dim(V_\gamma).
\]
Properties of \( \ell \)-weights

- The series
  \[
  \gamma_i^\pm(u) := \sum_{m=0}^{\infty} \gamma_{i,\pm m} u^{\pm m}
  \]
  are the Laurent expansions about 0, resp. \( \infty \), of a rational function \( \gamma_i(u) \) such that \( \gamma_i(0)\gamma_i(\infty) = 1 \).
- Such tuples of functions form an abelian group under multiplication.
- Moreover \((\gamma_i(u))_{i \in I}\) is in the subgroup generated by certain tuples of rational functions \( Y_{i,a} \) (and their inverses), given by
  \[
  (Y_{i,a})_i(u) := q_i \frac{1 - q_i^{-2} au}{1 - au}, \quad (Y_{i,a})_{j}(u) = 1, \; j \neq i.
  \]
Proposition

The $q$-character map $\chi_q$ is an injective homomorphism of rings

$$\chi_q : \text{Grothendieck ring of } C \longrightarrow \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times},$$

where $C$ is the category of finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$.

- so $\chi_q(V \otimes W) = \chi_q(V) \chi_q(W)$ and $\chi_q(V \oplus W) = \chi_q(V) + \chi_q(W)$.

$\chi_q$ is a refinement of the usual formal character map $\chi$:

if $Y_{i,a} \mapsto y_i := e^{\omega_i}$ then $\chi_q(V) \mapsto \chi(V)$.

[Frenkel & Reshetikhin]
There is a natural analogy:

\[ Y_{i,a} \leftrightarrow \text{fundamental weight } \omega_i \]

Products of \( Y_{i,a}^{\pm 1} \)'s \((i \in I, a \in \mathbb{C}^\times)\) \(\leftrightarrow\) Integral weights

Products of \( Y_{i,a} \)'s \((i \in I, a \in \mathbb{C}^\times)\) \(\leftrightarrow\) Dominant integral weights
Quantum-loop analogs of weights and roots

\[ Y_{i,a} \longleftrightarrow \text{fundamental weight } \omega_i \]

Products of \( Y_{i,a}^{\pm 1} \)'s \((i \in I, \ a \in \mathbb{C}^\times)\) \(\longleftrightarrow \) Integral weights

Products of \( Y_{i,a} \)'s \((i \in I, \ a \in \mathbb{C}^\times)\) \(\longleftrightarrow \) Dominant integral weights

\[ A_{i,a} \longleftrightarrow \text{Simple root } \alpha_i \]

Definition

Let \( A_{i,a} \) be the rational \( \ell \)-weight defined by

\[
(A_{j,a})_i(u) := \frac{q^{\pm B_{ij} - ua}}{1 - q^{\pm B_{ij} ua}}.
\]

Then

\[
\chi_q(L(\gamma)) \in \gamma \mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times},
\]

where \( L(\gamma) \) is the irreducible \( U_q(\hat{g}) \)-module with highest \( \ell \)-weight \( \gamma \).

[Frenkel & Mukhin]
Simplest Example: characters vs q-characters

Let $g = sl_2$. Then $e^{\alpha_1} = e^{2\omega_1}$, and $A_{1,a} = Y_{1,a} Y_{1,aq^2}$.
Quantum-loop analogs of weights and roots

\[ Y_{i,a} \leftrightarrow \text{fundamental weight } \omega_i \]

Products of \( Y_{i,a}^{\pm 1} \)'s \( (i \in I, a \in \mathbb{C}^\times) \) \( \leftrightarrow \) Integral weights

Products of \( Y_{i,a} \)'s \( (i \in I, a \in \mathbb{C}^\times) \) \( \leftrightarrow \) Dominant integral weights

\( A_{i,a} \leftrightarrow \text{Simple root } \alpha_i \)

However... the raising/lowering operators \( x_i^{\pm} \) of \( U_q(g) \) act with definite weight. i.e. for any weight module \( V \),

\[ v \in V_\omega \implies x_i^{\pm} v \in V_{\omega \pm \alpha_i}. \]

Can we find a linear combination of the modes \( x_{i,r}^{\pm} \) that acts with definite \( \ell \)-weight \( A_{i,a}^{\pm 1} \) in all representations? No.
Relevant defining relations are

\[ U_q(g) : \quad k_i x_j^\pm = q^{\pm B_{ij}} x_j^\pm k_i, \]

\[ U_q(L g) : \quad \phi_i^+(u) x_j^\pm(v) = \frac{q^{\pm B_{ij}} - uv}{1 - q^{\pm B_{ij}} uv} x_j^\pm(v) \phi_i^+(u), \]

...so, naively, just “set \( v = a \)”, for any \( a \in \mathbb{C}^\times \), and find

\[ \phi_i^+(u) x_j^\pm(a) = \frac{q^{\pm B_{ij}} - ua}{1 - q^{\pm B_{ij}} \cdot u a} x_j^\pm(a) \phi_i^+(u) \]

...except that this is nonsense:

\[ x_i^\pm(z) := \sum_{r \in \mathbb{Z}} z^{-r} x_{i,r}^\pm \] is a formal series in \( z \), and

\[ x_i^\pm(a) = \sum_{r \in \mathbb{Z}} a^{-r} x_{i,r}^\pm \] with \( a \in \mathbb{C}^\times \) is ill-defined in a sense that is not merely technical: its would-be matrix representatives have divergent entries.
Suppose \( \rho : U_q(\mathcal{L} \mathfrak{g}) \to \text{End}(V) \) is any finite-dimensional representation of \( U_q(\mathcal{L} \mathfrak{g}) \).

Let \( \mathcal{P} \subset \mathbb{C}^\times \) be the finite set of points where the \( \ell \)-weights of \( V \) have poles.

Set \( M := 2 \times (\text{max dimension of any } \ell \text{-weight space of } V) \).

**Proposition**

There exist linear maps \( E_{i,a,m}^\pm \) and \( H_{i,a,m} \) in \( \text{End}(V) \) such that for each \( i \in I \),

\[
\rho(x_i^\pm(z)) = \sum_{a \in \mathcal{P}} \sum_{m=0}^{M} E_{i,a,m}^\pm \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right),
\]

\[
\rho \left( \frac{\phi_i^+(1/z) - \phi_i^-(1/z)}{q_i - q_i^{-1}} \right) = \sum_{a \in \mathcal{P}} \sum_{m=0}^{M} H_{i,a,m} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right).
\]

Moreover, \( E_{i,a,m}^\pm(V_\mu) \subseteq V_{\mu A_{i,a}^{\pm 1}} \) and \( H_{i,a,m}(V_\mu) \subseteq V_\mu \) for each \( \ell \)-weight \( \mu \).

**Question:** what are the algebraic relations obeyed by these maps \( E_{i,a,m}^\pm \) and \( H_{i,a,m} \)?
First look at the maps $E_{i,a,m}^\pm$. Regard them as abstract generators of some algebra $A^\pm$.

Question: what does it take for the map

$$U^\pm \xrightarrow{\theta} A^\pm$$

$$x_i^\pm(z) \mapsto \sum_{a \in \mathcal{P}} \sum_{m=0}^{\infty} E_{i,a,m}^\pm \frac{a_m^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right)$$

to be a homomorphism of algebras?

($\exists$ natural decreasing filtration of $A^\pm$ in which $E_{i,a,m}^\pm \in \mathcal{F}_m(A^\pm)$. We work in the completion w.r.t. this filtration, so $\sum_{m=0}^{\infty}$ is well defined here.)

Have quadratic defining relation of $U^\pm$:

$$(u - q^\pm B_{ij} v) x_i^\pm(u) x_j^\pm(v) = (q^\pm B_{ij} u - v) x_j^\pm(v) x_i^\pm(u),$$

Sufficient to impose

$$(a - bq^\pm B_{ij}) E_{i,a,m}^\pm E_{j,b,n}^\pm + a E_{i,a,m+1}^\pm E_{j,b,n}^\pm - bq^\pm B_{ij} E_{i,a,m}^\pm E_{j,b,n+1}^\pm$$

$$= (aq^\pm B_{ij} - b) E_{j,b,n}^\pm E_{i,a,m}^\pm + aq^\pm B_{ij} E_{j,b,n}^\pm E_{i,a,m+1}^\pm - b E_{j,b,n+1}^\pm E_{i,a,m}^\pm.$$
\[
(a - bq^{\pm B_{ij}})E_{i,a,m}^\pm E_{j,b,n}^\pm + aE_{i,a,m+1}^\pm E_{j,b,n}^\pm - bq^{\pm B_{ij}} E_{i,a,m}^\pm E_{j,b,n+1}^\pm = (aq^{\pm B_{ij}} - b)E_{j,b,n}^\pm E_{i,a,m}^\pm + aq^{\pm B_{ij}} E_{j,b,n}^\pm E_{i,a,m+1}^\pm - bE_{j,b,n+1}^\pm E_{i,a,m}^\pm
\]

Define
\[
E_{i,a}^\pm(z) := \sum_{m=0}^{\infty} \frac{z^m}{m!} E_{i,a,m}^\pm \in \mathcal{A}^{\pm}[z].
\]

Then

**Proposition**

Whenever \( a \neq bq^{\pm B_{ij}} \),

\[
E_{i,a}^\pm(z)E_{j,b}^\pm(w) = \frac{aq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial z} \right) - b \left( 1 + \frac{\partial}{\partial w} \right)}{a \left( 1 + \frac{\partial}{\partial z} \right) - bq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial w} \right)} E_{j,b}^\pm(w)E_{i,a}^\pm(z)
\]

If \( a = bq^{\pm B_{ij}} \), say \( E_{i,a}^\pm(z) \) sticks to the left of \( E_{j,b}^\pm(w) \).
Also have the **Serre relations** in $U_q(\mathcal{Lg})$:

$$\sum_{\pi \in \Sigma_s} \sum_{r=0}^{s} (-1)^r \begin{bmatrix} s \\ r \end{bmatrix} q_i \ x_i^\pm (w_{\pi(1)}) \cdots x_i^\pm (w_{\pi(r)}) x_j^\pm (z) x_i^\pm (w_{\pi(r+1)}) \cdots x_i^\pm (w_{\pi(s)}) = 0,$$

for all $i \neq j$, where $s = 1 - C_{ij}$ and $\Sigma_s$ is the symmetric group on $s$ letters.

Again, we want

$$x_i^\pm (z) \mapsto \sum_{a \in P} \sum_{m=0}^{\infty} E_{i,a,m}^\pm a^m \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right)$$

to be a homomorphism.

It is sufficient to impose:

$$\sum_{\pi \in \Sigma_s} \sum_{r=0}^{s} (-1)^r \begin{bmatrix} s \\ r \end{bmatrix} E_{i,a_{\pi(1)}^\pm}^\pm (z_{\pi(1)}) \cdots E_{i,a_{\pi(r)}^\pm}^\pm (z_{\pi(r)}) E_{j,b}^\pm (w) E_{i,a_{\pi(r+1)}^\pm}^\pm (z_{\pi(r+1)}) \cdots E_{i,a_{\pi(s)}^\pm}^\pm (z_{\pi(s)}) = 0$$

But find this is almost always **automatically** true, given the quadratic $E^\pm E^\pm$ relations, by virtue of a certain combinatorial identity due to [Jing].

what is left is...
Serre relations in $A^\pm$

\[ 0 = E_{j,b,n}^+ E_{i,bq^{-1},m_1}^+ E_{i,bq,m_2}^+ - [2] q E_{i,bq,m_2}^+ E_{j,b,n}^+ E_{i,bq^{-1},m_1}^+ \]
\[ + E_{i,bq^{-1},m_1}^+ E_{i,bq,m_2}^+ E_{j,b,n}^+ \]

\[ 0 = E_{j,b,n}^- E_{i,bq^{-2},m_1}^- E_{i,bq^2,m_2}^- - [3] q E_{i,bq^2,m_3}^- E_{j,b,n}^- E_{i,bq^{-2},m_1}^- \]
\[ + [3] q E_{i,b,m_2}^- E_{i,bq^2,m_3}^- E_{j,b,n}^- E_{i,bq^{-2},m_1}^- - E_{i,bq^{-2},m_1}^- E_{i,b,m_2}^- E_{i,bq^2,m_3}^- E_{j,b,n}^- \]
\[ 0 = E_{i,b,n}^- E_{j,bq^{-2},m_1}^- E_{j,bq^2,m_2}^- - [2] q^2 E_{j,bq^2,m_2}^- E_{i,b,n}^- E_{j,bq^{-2},m_1}^- + E_{j,bq^{-2},m_1}^- E_{i,bq^2,m_2}^- E_{j,b,n}^- \]

\[ 0 = E_{j,b,n}^- E_{i,bq^{-3},m_1}^- E_{i,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- - [4] q E_{i,bq^3,m_4}^- E_{j,b,n}^- E_{i,bq^{-3},m_1}^- E_{i,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- \]
\[ + \binom{4}{2} q E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- E_{j,b,n}^- E_{i,bq^{-3},m_1}^- E_{i,bq^{-1},m_2}^- - [4] q E_{i,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- E_{j,b,n}^- \]
\[ + E_{j,bq^{-3},m_1}^- E_{j,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- E_{j,b,n}^- \]
\[ 0 = E_{i,b,n}^- E_{j,bq^{-3},m_1}^- E_{j,bq^3,m_2}^- - [2] q^3 E_{j,bq^3,m_2}^- E_{i,b,n}^- E_{j,bq^{-3},m_1}^- + E_{j,bq^{-3},m_1}^- E_{i,bq^3,m_2}^- E_{j,b,n}^- \]
Definition

Let \( \mathcal{A} \) be the associative unital algebra over \( \mathbb{C} \) generated by

\[
K_{i}^{\pm 1}, \quad E_{i,a,m}, \quad H_{i,a,m}, \quad i \in I, \ a \in \mathcal{P}, \ m \in \mathbb{Z}_{\geq 0},
\]

subject to the following relations. For all \( i, j \in I, \ a, b \in \mathcal{P}, \) and \( m, n \in \mathbb{Z}_{\geq 0}, \)

\[
K_{i}K_{i}^{-1} = 1, \quad K_{i}E_{j,b,m} = E_{j,b,m}q_{B_{ij}}^{\pm}K_{i}, \quad K_{i}K_{j} = K_{j}K_{i}
\]

\[
K_{i}H_{j,b,m} = H_{j,b,m}K_{i}, \quad H_{i,a,m}H_{j,b,n} = H_{j,b,n}H_{i,a,m}
\]

\[
[E_{i,a,m}, E_{j,b,n}] = \delta_{i j} \delta_{a b} H_{i,a,m+n}
\]

\[
(a - bq_{B_{ij}}^{\pm})E_{i,a,m}E_{j,b,n} + aE_{i,a,m+1}E_{j,b,n} - bq_{B_{ij}}^{\pm}E_{i,a,m}E_{j,b,n+1} = (aq_{B_{ij}}^{\pm} - b)E_{j,b,n}E_{i,a,m} + aq_{B_{ij}}^{\pm}E_{j,b,n+1}E_{i,a,m+1} - bE_{j,b,n+1}E_{i,a,m}
\]

\[
(a - bq_{B_{ij}}^{\pm})H_{i,a,m}E_{j,b,n} + aH_{i,a,m+1}E_{j,b,n} - bq_{B_{ij}}^{\pm}H_{i,a,m}E_{j,b,n+1} = (aq_{B_{ij}}^{\pm} - b)E_{j,b,n}H_{i,a,m} + aq_{B_{ij}}^{\pm}E_{j,b,n+1}H_{i,a,m+1} - bE_{j,b,n+1}H_{i,a,m}
\]

\[
\sum_{a \in \mathcal{P}} H_{i,a,0} = \frac{K_{i} - K_{i}^{-1}}{q_{i} - q_{i}^{-1}}.
\]

and the Serre relations shown before.
Main result

Theorem

There is a homomorphism of algebras \( \theta : U_q(\mathcal{L}g) \to A \) defined by \( k_i \mapsto K_i \) and

\[
E_{i,a,m}^\pm \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right),
\]

\[
\frac{\phi_i^+(1/z) - \phi_i^-(1/z)}{q_i - q_i^{-1}} \mapsto \sum_{a \in \mathcal{P}} \sum_{m \in \mathbb{Z}_{\geq 0}} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right).
\]

Moreover, let \( C_\mathcal{P} \) be the category of those finite-dimensional representations of \( U_q(\mathcal{L}g) \) whose \( \ell \)-weights have poles at points in \( \mathcal{P} \subset \mathbb{C}^\times \) only. Then every \( V \in \text{Ob}(C_\mathcal{P}) \) is the pull-back by \( \theta \) of a finite-dimensional representation of \( A \).
Idea is to use this result to construct explicit $\ell$-weight bases of $U_q(\mathcal{L}g)$-modules.

**Example (in type $A_1$, the module $L(Y_0 Y_2^2)$)**

$$K v = q^3 v, \quad H_{0,0} v = (q^2 + 1 + q^{-2}) v, \quad H_{0,1} v = (q^2 - q^{-2}) v.$$  

Shorthand $E_{-k,m} := E_{1,aq^k,m}$, $H_{k,m} := H_{1,aq^k,m}$, $Y_k := Y_{1,aq^k}$, $A_k := A_{1,aq^k}$.
\[ L(Y_0^3 Y_2^2) : K \nu = q^5 \nu, \quad H_{0,0} \nu = \nu, \quad H_{2,0} \nu = (q^4 + q^2 + q^{-2} + q^{-4}) \nu, \quad H_{2,1} \nu = (q^4 + 2q^2 - 2q^{-2} - q^{-4}) \nu. \]
Outlook

- Basis for $\mathcal{A}$?
  Easy to check that $\mathcal{A} = \mathcal{A}^- \cdot \mathcal{A}^0 \cdot \mathcal{A}^+$.
- Basis for $\mathcal{A}^\pm$? A canonical basis for $\mathcal{A}^\pm$?
- Coproduct? From Drinfeld “current” coproduct?
- Rational limit à la Yangians.
- All for $\mathfrak{g}$ semisimple. Does it work for all $\mathfrak{g}$ with symmetrizable Cartan matrix? (affinize $U_q(\mathfrak{g})$ with $\mathfrak{g}$ affine $\rightsquigarrow$ one approach to quantum toroidal algebras)
- (in simply-laced types)
  Relation to geometric realizations of representations?