Quantum Affine Algebras and ℓ-root operators

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Quantum Affine Algebras: loop perspective

$$U_q(\widehat{\mathfrak{g}})/(c-1) \cong U_q(\mathcal{L}\mathfrak{g})$$

- $\mathfrak{g}$: simple Lie algebra over $\mathbb{C}$.
- $i \in I$: nodes of Dynkin diagram of $\mathfrak{g}$.
- $q \in \mathbb{C}^\times$, transcendental.
- $\mathcal{L}\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$: loop algebra of $\mathfrak{g}$.

Generators of quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ in Drinfeld's current presentation:

$x_i^-(z)$	, $\phi_i^\pm(z)$,	$x_i^+(z)$,
lowering	"Cartan"	raising

Theorem [Beck] There is a triangular decomposition

$$U_q(\mathcal{L}\mathfrak{g}) \quad \cong_{\mathbb{C}} \quad U^- \quad \otimes \quad U^0 \quad \otimes \quad U^+.$$

Theorem [Chari-Pressley] Every finite-dimensional $U_q(\mathcal{L}\mathfrak{g})$-module is "highest weight" *with respect to this triangular decomposition*.

ℓ-weights and q-characters

$$\underset{\text{lowering}}{x_i^-(z)} \quad , \quad \underset{\text{“Cartan”}}{\phi_i^\pm(z)}, \qquad \underset{\text{raising}}{x_i^+(z)},$$

$$U_q(\mathcal{L}\mathfrak{g}) \qquad \cong_{\mathbb{C}} \qquad U^- \quad \otimes \quad U^0 \quad \otimes \quad U^+.$$

Eigenvalues of $\phi_i^\pm(z) = \sum_{r=0}^{\infty} \phi_{i,\pm r}^\pm z^{\pm r}$ are called ℓ-weights.

Let V be a $U_q(\mathcal{L}\mathfrak{g})$-module of finite dimension.

Define the *ℓ-weight space* V_γ by

$$V_\gamma := \{v \in V : \exists k \in \mathbb{N},\ (\phi_{i,\pm m}^\pm - \gamma_{i,\pm m}^\pm)^k v = 0\ \forall i \in I, m \geq 0\}.$$

Have ℓ-weight decomposition and *“q-character”* [Frenkel & Reshetikhin]

$$V = \bigoplus_\gamma V_\gamma\ , \qquad \chi_q(V) \underset{\substack{\text{formal}\\ \text{sum}}}{:=} \sum_\gamma \gamma \dim(V_\gamma).$$

Properties of ℓ-weights

- The series
$$\gamma_i^{\pm}(u) := \sum_{m=0}^{\infty} \gamma_{i,\pm m}^{\pm} u^{\pm m}$$
are the Laurent expansions about 0, resp. ∞, of a rational function $\gamma_i(u)$ such that $\gamma_i(0)\gamma_i(\infty) = 1$.
- Such tuples of functions form an abelian group under multiplication.
- Moreover $(\gamma_i(u))_{i\in I}$ is in the subgroup generated by certain tuples of rational functions $Y_{i,a}$ (and their inverses), given by
$$(Y_{i,a})_i(u) := q_i \frac{1 - q_i^{-2} au}{1 - au}, \qquad (Y_{i,a})_j(u) = 1,\ j \neq i.$$

Proposition

The q-character map χ_q is an injective homomorphism of rings

$$\chi_q : \textit{Grothendieck ring of } \mathcal{C} \longrightarrow \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times},$$

where $\mathcal{C}$ is the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$.

– *so* $\chi_q(V \otimes W) = \chi_q(V)\chi_q(W)$ *and* $\chi_q(V \oplus W) = \chi_q(V) + \chi_q(W)$.

χ_q is a refinement of the usual formal character map χ:

$$\textit{if} \quad Y_{i,a} \mapsto y_i := e^{\omega_i} \quad \textit{then} \quad \chi_q(V) \longmapsto \chi(V).$$

[Frenkel & Reshetikhin]

There is a natural analogy:

$$
\begin{array}{rcl}
Y_{i,a} & \longleftrightarrow & \text{fundamental weight } \omega_i \\
\text{Products of } Y_{i,a}^{\pm 1}\text{'s } (i \in I,\ a \in \mathbb{C}^\times) & \longleftrightarrow & \text{Integral weights} \\
\text{Products of } Y_{i,a}\text{'s } (i \in I,\ a \in Cx) & \longleftrightarrow & \text{Dominant integral weights}
\end{array}
$$

Quantum-loop analogs of weights and roots

$Y_{i,a}$	$\longleftrightarrow$	fundamental weight ω_i
Products of $Y_{i,a}^{\pm 1}$'s ($i \in I$, $a \in \mathbb{C}^\times$)	$\longleftrightarrow$	Integral weights
Products of $Y_{i,a}$'s ($i \in I$, $a \in \mathbb{C}^\times$)	$\longleftrightarrow$	Dominant integral weights
$A_{i,a}$	$\longleftrightarrow$	Simple root α_i

Definition

Let $A_{i,a}$ be the rational ℓ-weight defined by $(A_{j,a})_i(u) := \frac{q^{\pm B_{ij} - ua}}{1 - q^{\pm B_{ij}} ua}$.

Then

$$\chi_q(L(\gamma)) \in \gamma\, \mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times},$$

where $L(\gamma)$ is the irreducible $U_q(\widehat{\mathfrak{g}})$-module with highest ℓ-weight γ.

[Frenkel & Mukhin]

Simplest Example: characters vs q-characters

Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $e^{\alpha_1} = e^{2\omega_1}$, and $A_{1,a} = Y_{1,a} Y_{1,aq^2}$.

$\chi(V_{2\omega_1})$

$$\begin{tikzcd}
e^{2\omega_1} \arrow[d, "e^{-\alpha_1}"'] \\
1 \arrow[d, "e^{-\alpha_1}"'] \\
e^{-2\omega_1}
\end{tikzcd}$$

$\chi_q(L(Y_{1,a} Y_{1,b}))$

$$\begin{tikzcd}
 & Y_{1,a} Y_{1,b} \arrow[dl, "A_{1,b}^{-1}"'] \arrow[dr, "A_{1,a}^{-1}"] & \\
Y_{1,a} Y_{1,bq^2}^{-1} \arrow[dr, "A_{1,a}^{-1}"'] & & Y_{1,aq^2}^{-1} Y_{1,b} \arrow[dl, "A_{1,b}^{-1}"] \\
 & Y_{1,aq^2}^{-1} Y_{1,bq^2}^{-1}. &
\end{tikzcd}$$

Quantum-loop analogs of weights and roots

$$
\begin{array}{rcl}
Y_{i,a} & \longleftrightarrow & \text{fundamental weight } \omega_i \\
\text{Products of } Y_{i,a}^{\pm 1}\text{'s } (i \in I,\ a \in \mathbb{C}^\times) & \longleftrightarrow & \text{Integral weights} \\
\text{Products of } Y_{i,a}\text{'s } (i \in I,\ a \in \mathbb{C}^\times) & \longleftrightarrow & \text{Dominant integral weights} \\
A_{i,a} & \longleftrightarrow & \text{Simple root } \alpha_i
\end{array}
$$

However...
the raising/lowering operators $x_i^\pm$ of $U_q(\mathfrak{g})$ act with definite weight.
i.e. for any weight module V,

$$v \in V_\omega \implies x_i^\pm v \in V_{\omega \pm \alpha_i}.$$

Can we find a linear combination of the modes $x_{i,r}^\pm$ that acts with definite ℓ-weight $A_{i,a}^{\pm 1}$ in all representations? No.

Relevant defining relations are

$$U_q(\mathfrak{g}): \qquad k_i x_j^{\pm} = q^{\pm B_{ij}} x_j^{\pm} k_i,$$

$$U_q(\mathcal{L}\mathfrak{g}): \qquad \phi_i^+(u)\, x_j^{\pm}(v) = \frac{q^{\pm B_{ij}} - uv}{1 - q^{\pm B_{ij}} uv}\, x_j^{\pm}(v)\, \phi_i^+(u),$$

...so, naively, just "set $v = a$", for any $a \in \mathbb{C}^\times$, and find

$$\text{“} \quad \phi_i^+(u)\, x_j^{\pm}(a) = \underbrace{\frac{q^{\pm B_{ij}} - ua}{1 - q^{\pm B_{ij}} ua}}_{(A_{j,a})_i(u)}\, x_j^{\pm}(a)\, \phi_i^+(u) \quad \text{”}$$

...except that this is nonsense:

- $x_i^{\pm}(z) := \sum_{r \in \mathbb{Z}} z^{-r} x_{i,r}^{\pm}$ is a formal series in z, and
- $x_i^{\pm}(a) = \sum_{r \in \mathbb{Z}} a^{-r} x_{i,r}^{\pm}$ with $a \in \mathbb{C}^\times$ is ill-defined in a sense that is not merely technical: its would-be matrix representatives have divergent entries.

Suppose $\rho : U_q(\mathcal{L}\mathfrak{g}) \to \mathrm{End}(V)$ is any finite-dimensional representation of $U_q(\mathcal{L}\mathfrak{g})$.

Let $\mathcal{P} \subset \mathbb{C}^\times$ be the finite set of points where the ℓ-weights of V have poles.

Set $M := 2 \times$ (max dimension of any ℓ-weight space of V).

Proposition

There exist linear maps $E^{\pm}_{i,a,m}$ and $H_{i,a,m}$ in $\mathrm{End}(V)$ such that for each $i \in I$,

$$\rho(x_i^{\pm}(z)) = \sum_{a \in \mathcal{P}} \sum_{m=0}^{M} E^{\pm}_{i,a,m} \frac{a^m}{m!} \left(\frac{\partial}{\partial a}\right)^m \delta\left(\frac{a}{z}\right),$$

$$\rho\left(\frac{\phi_i^+(1/z) - \phi_i^-(1/z)}{q_i - q_i^{-1}}\right) = \sum_{a \in \mathcal{P}} \sum_{m=0}^{M} H_{i,a,m} \frac{a^m}{m!} \left(\frac{\partial}{\partial a}\right)^m \delta\left(\frac{a}{z}\right).$$

Moreover, $E^{\pm}_{i,a,m}(V_{\boldsymbol{\mu}}) \subseteq V_{\boldsymbol{\mu} A^{\pm 1}_{i,a}}$ and $H_{i,a,m}(V_{\boldsymbol{\mu}}) \subseteq V_{\boldsymbol{\mu}}$ for each ℓ-weight $\boldsymbol{\mu}$.

Question: what are the algebraic relations obeyed by these maps $E^{\pm}_{i,a,m}$ and $H_{i,a,m}$?

- First look at the maps $\mathsf{E}^{\pm}_{i,a,m}$.
  Regard them as abstract generators of some algebra $\mathcal{A}^{\pm}$
- Question: what does it take for the map

$$\begin{aligned} U^{\pm} &\xrightarrow[\theta]{} \mathcal{A}^{\pm} \\ x_i^{\pm}(z) &\longmapsto \sum_{a\in\mathcal{P}}\sum_{m=0}^{\infty} \mathsf{E}^{\pm}_{i,a,m}\frac{a^m}{m!}\left(\frac{\partial}{\partial a}\right)^m \delta\left(\frac{a}{z}\right) \end{aligned}$$

  to be a homomorphism of algebras?
  ($\exists$ natural decreasing filtration of $\mathcal{A}^{\pm}$ in which $\mathsf{E}^{\pm}_{i,a,m} \in \mathcal{F}_m(\mathcal{A}^{\pm})$. We work in the completion w.r.t. this filtration, so $\sum_{m=0}^{\infty}$ is well defined here.)
- Have quadratic defining relation of $U^{\pm}$:

$$\left(u - q^{\pm B_{ij}} v\right) x_i^{\pm}(u)\, x_j^{\pm}(v) \;=\; \left(q^{\pm B_{ij}} u - v\right)\, x_j^{\pm}(v)\, x_i^{\pm}(u),$$

- Sufficient to impose

$$\begin{aligned} & (a - bq^{\pm B_{ij}})\mathsf{E}^{\pm}_{i,a,m}\mathsf{E}^{\pm}_{j,b,n} + a\mathsf{E}^{\pm}_{i,a,m+1}\mathsf{E}^{\pm}_{j,b,n} - bq^{\pm B_{ij}}\mathsf{E}^{\pm}_{i,a,m}\mathsf{E}^{\pm}_{j,b,n+1} \\ = \;& (aq^{\pm B_{ij}} - b)\mathsf{E}^{\pm}_{j,b,n}\mathsf{E}^{\pm}_{i,a,m} + aq^{\pm B_{ij}}\mathsf{E}^{\pm}_{j,b,n}\mathsf{E}^{\pm}_{i,a,m+1} - b\mathsf{E}^{\pm}_{j,b,n+1}\mathsf{E}^{\pm}_{i,a,m} \end{aligned}$$

$$
\begin{aligned}
& (a - bq^{\pm B_{ij}})\mathrm{E}^{\pm}_{i,a,m}\mathrm{E}^{\pm}_{j,b,n} + a\mathrm{E}^{\pm}_{i,a,m+1}\mathrm{E}^{\pm}_{j,b,n} - bq^{\pm B_{ij}}\mathrm{E}^{\pm}_{i,a,m}\mathrm{E}^{\pm}_{j,b,n+1} \\
= \quad & (aq^{\pm B_{ij}} - b)\mathrm{E}^{\pm}_{j,b,n}\mathrm{E}^{\pm}_{i,a,m} + aq^{\pm B_{ij}}\mathrm{E}^{\pm}_{j,b,n}\mathrm{E}^{\pm}_{i,a,m+1} - b\mathrm{E}^{\pm}_{j,b,n+1}\mathrm{E}^{\pm}_{i,a,m}
\end{aligned}
$$

Define

$$
\mathrm{E}^{\pm}_{i,a}(z) := \sum_{m=0}^{\infty} \frac{z^m}{m!}\mathrm{E}^{\pm}_{i,a,m} \in \mathcal{A}^{\pm}[[z]].
$$

Then

Proposition

Whenever $a \neq bq^{\pm B_{ij}}$,

$$
\mathrm{E}^{\pm}_{i,a}(z)\mathrm{E}^{\pm}_{j,b}(w) = \frac{aq^{\pm B_{ij}}\left(1 + \frac{\partial}{\partial z}\right) - b\left(1 + \frac{\partial}{\partial w}\right)}{a\left(1 + \frac{\partial}{\partial z}\right) - bq^{\pm B_{ij}}\left(1 + \frac{\partial}{\partial w}\right)}\mathrm{E}^{\pm}_{j,b}(w)\mathrm{E}^{\pm}_{i,a}(z)
$$

If $a = bq^{\pm B_{ij}}$, say $\mathrm{E}^{\pm}_{i,a}(z)$ *sticks to the left of* $\mathrm{E}^{\pm}_{j,b}(w)$.

- Also have the Serre relations in $U_q(\mathcal{L}\mathfrak{g})$:

$$\sum_{\pi\in\Sigma_s}\sum_{r=0}^{s}(-1)^r\begin{bmatrix}s\\r\end{bmatrix}_{q_i} x_i^{\pm}(w_{\pi(1)})\dots x_i^{\pm}(w_{\pi(r)})x_j^{\pm}(z)x_i^{\pm}(w_{\pi(r+1)})\dots x_i^{\pm}(w_{\pi(s)})=0,$$

  for all $i\neq j$, where $s=1-C_{ij}$ and Σ_s is the symmetric group on s letters.
- Again, we want

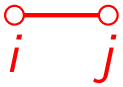
$$x_i^{\pm}(z)\underset{\theta}{\mapsto}\sum_{a\in\mathcal{P}}\sum_{m=0}^{\infty}\mathrm{E}^{\pm}_{i,a,m}\frac{a^m}{m!}\left(\frac{\partial}{\partial a}\right)^m\delta\left(\frac{a}{z}\right)$$

  to be a homomorphism.
- It is sufficient to impose:

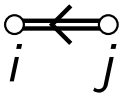
$$\sum_{\pi\in\Sigma_s}\sum_{r=0}^{s}(-1)^r\begin{bmatrix}s\\r\end{bmatrix}_{q_i}\mathrm{E}^{\pm}_{i,a_{\pi(1)}}(z_{\pi(1)})\dots\mathrm{E}^{\pm}_{i,a_{\pi(r)}}(z_{\pi(r)})\mathrm{E}^{\pm}_{j,b}(w)\mathrm{E}^{\pm}_{i,a_{\pi(r+1)}}(z_{\pi(r+1)})\dots\mathrm{E}^{\pm}_{i,a_{\pi(s)}}(z_{\pi(s)})=0$$

- But find this is almost always automatically true, given the quadratic $\mathrm{E}^{\pm}\mathrm{E}^{\pm}$ relations, by virtue of a certain combinatorial identity due to [Jing].
- what is left is...

Serre relations in $\mathcal{A}^{\pm}$

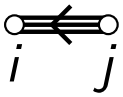


$$0 = \mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-1},m_1}\mathsf{E}^-_{i,bq,m_2} - [2]_q\mathsf{E}^-_{i,bq,m_2}\mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-1},m_1} + \mathsf{E}^-_{i,bq^{-1},m_1}\mathsf{E}^-_{i,bq,m_2}\mathsf{E}^-_{j,b,n}$$



$$0=\mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-2},m_1}\mathsf{E}^-_{i,b,m_2}\mathsf{E}^-_{i,bq^2,m_3} - [3]_q\mathsf{E}^-_{i,bq^2,m_3}\mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-2},m_1}\mathsf{E}^-_{i,b,m_2} + [3]_q\mathsf{E}^-_{i,b,m_2}\mathsf{E}^-_{i,bq^2,m_3}\mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-2},m_1} - \mathsf{E}^-_{i,bq^{-2},m_1}\mathsf{E}^-_{i,b,m_2}\mathsf{E}^-_{i,bq^2,m_3}\mathsf{E}^-_{j,b,n}$$

$$0=\mathsf{E}^-_{i,b,n}\mathsf{E}^-_{j,bq^{-2},m_1}\mathsf{E}^-_{j,bq^2,m_2} - [2]_{q^2}\mathsf{E}^-_{j,bq^2,m_2}\mathsf{E}^-_{i,b,n}\mathsf{E}^-_{j,bq^{-2},m_1} + \mathsf{E}^-_{i,bq^{-2},m_1}\mathsf{E}^-_{i,bq^2,m_2}\mathsf{E}^-_{j,b,n}$$



$$0=\mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-3},m_1}\mathsf{E}^-_{i,bq^{-1},m_2}\mathsf{E}^-_{i,bq^1,m_3}\mathsf{E}^-_{i,bq^3,m_4} - [4]_q\mathsf{E}^-_{i,bq^3,m_4}\mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-3},m_1}\mathsf{E}^-_{i,bq^{-1},m_2}\mathsf{E}^-_{i,bq^1,m_3}$$

$$+\binom{4}{2}_q\mathsf{E}^-_{i,bq^1,m_3}\mathsf{E}^-_{i,bq^3,m_4}\mathsf{E}^-_{j,b,n}\mathsf{E}^-_{i,bq^{-3},m_1}\mathsf{E}^-_{i,bq^{-1},m_2} - [4]_q\mathsf{E}^-_{i,bq^{-1},m_2}\mathsf{E}^-_{i,bq^1,m_3}\mathsf{E}^-_{i,bq^3,m_4}\mathsf{E}^-_{j,b,n}\mathsf{E}$$

$$+\mathsf{E}^-_{i,bq^{-3},m_1}\mathsf{E}^-_{i,bq^{-1},m_2}\mathsf{E}^-_{i,bq^1,m_3}\mathsf{E}^-_{i,bq^3,m_4}\mathsf{E}^-_{j,b,n}$$

$$0=\mathsf{E}^-_{i,b,n}\mathsf{E}^-_{j,bq^{-3},m_1}\mathsf{E}^-_{j,bq^3,m_2} - [2]_{q^3}\mathsf{E}^-_{j,bq^3,m_2}\mathsf{E}^-_{i,b,n}\mathsf{E}^-_{j,bq^{-3},m_1} + \mathsf{E}^-_{i,bq^{-3},m_1}\mathsf{E}^-_{i,bq^3,m_2}\mathsf{E}^-_{j,b,n}.$$

Definition

Let $\mathcal{A}$ be the associative unital algebra over $\mathbb{C}$ generated by

$$\mathrm{K}_i^{\pm 1}, \quad \mathrm{E}_{i,a,m}^{\pm}, \quad \mathrm{H}_{i,a,m}, \quad i \in I,\ a \in \mathcal{P},\ m \in \mathbb{Z}_{\geq 0}, \tag{1}$$

subject to the following relations. For all $i, j \in I$, $a, b \in \mathcal{P}$, and $m, n \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{K}_i \mathrm{K}_i^{-1} = 1, \quad \mathrm{K}_i \mathrm{E}_{j,b,m}^{\pm} = \mathrm{E}_{j,b,m}^{\pm} q^{\pm B_{i\,j}} \mathrm{K}_i, \quad \mathrm{K}_i \mathrm{K}_j = \mathrm{K}_j \mathrm{K}_i \tag{2}$$

$$\mathrm{K}_i \mathrm{H}_{j,b,m} = \mathrm{H}_{j,b,m} \mathrm{K}_i, \quad \mathrm{H}_{i,a,m} \mathrm{H}_{j,b,n} = \mathrm{H}_{j,b,n} \mathrm{H}_{i,a,m} \tag{3}$$

$$\left[\mathrm{E}_{i,a,m}^{+}, \mathrm{E}_{j,b,n}^{-}\right] = \delta_{i\,j} \delta_{a,b} \mathrm{H}_{i,a,m+n} \tag{4}$$

$$\begin{aligned} & (a - bq^{\pm B_{i\,j}}) \mathrm{E}_{i,a,m}^{\pm} \mathrm{E}_{j,b,n}^{\pm} + a \mathrm{E}_{i,a,m+1}^{\pm} \mathrm{E}_{j,b,n}^{\pm} - bq^{\pm B_{i\,j}} \mathrm{E}_{i,a,m}^{\pm} \mathrm{E}_{j,b,n+1}^{\pm} \\ = \quad & (aq^{\pm B_{i\,j}} - b) \mathrm{E}_{j,b,n}^{\pm} \mathrm{E}_{i,a,m}^{\pm} + aq^{\pm B_{i\,j}} \mathrm{E}_{j,b,n}^{\pm} \mathrm{E}_{i,a,m+1}^{\pm} - b \mathrm{E}_{j,b,n+1}^{\pm} \mathrm{E}_{i,a,m}^{\pm} \end{aligned} \tag{5}$$

$$\begin{aligned} & (a - bq^{\pm B_{i\,j}}) \mathrm{H}_{i,a,m} \mathrm{E}_{j,b,n}^{\pm} + a \mathrm{H}_{i,a,m+1} \mathrm{E}_{j,b,n}^{\pm} - bq^{\pm B_{i\,j}} \mathrm{H}_{i,a,m} \mathrm{E}_{j,b,n+1}^{\pm} \\ = \quad & (aq^{\pm B_{i\,j}} - b) \mathrm{E}_{j,b,n}^{\pm} \mathrm{H}_{i,a,m} + aq^{\pm B_{i\,j}} \mathrm{E}_{j,b,n}^{\pm} \mathrm{H}_{i,a,m+1} - b \mathrm{E}_{j,b,n+1}^{\pm} \mathrm{H}_{i,a,m} \end{aligned} \tag{6}$$

$$\sum_{a \in \mathcal{P}} \mathrm{H}_{i,a,0} = \frac{\mathrm{K}_i - \mathrm{K}_i^{-1}}{q_i - q_i^{-1}}. \tag{7}$$

and the Serre relations shown before.

Main result

Theorem

There is a homomorphism of algebras $\theta : U_q(\mathcal{L}\mathfrak{g}) \to \mathcal{A}$ defined by $k_i \mapsto \mathsf{K}_i$ and

$$x_i^{\pm}(z) \mapsto \sum_{a \in \mathcal{P}} \sum_{m \in \mathbb{Z}_{\geq 0}} \mathsf{E}^{\pm}_{i,a,m} \frac{a^m}{m!} \left(\frac{\partial}{\partial a} \right)^m \delta\left(\frac{a}{z} \right),$$

$$\frac{\phi_i^+(1/z) - \phi_i^-(1/z)}{q_i - q_i^{-1}} \mapsto \sum_{a \in \mathcal{P}} \sum_{m \in \mathbb{Z}_{\geq 0}} \mathsf{H}_{i,a,m} \frac{a^m}{m!} \left(\frac{\partial}{\partial a} \right)^m \delta\left(\frac{a}{z} \right).$$

Moreover, let $\mathcal{C}_{\mathcal{P}}$ be the category of those finite-dimensional representations of $U_q(\mathcal{L}\mathfrak{g})$ whose ℓ-weights have poles at points in $\mathcal{P} \subset \mathbb{C}^\times$ only. Then every $V \in \mathrm{Ob}(\mathcal{C}_{\mathcal{P}})$ is the pull-back by θ of a finite-dimensional representation of $\mathcal{A}$.

Idea is to use this result to construct explicit ℓ-weight bases of $U_q(\mathcal{L}\mathfrak{g})$-modules.

Example (in type A_1, the module $L(Y_0 Y_2^2)$)

$\mathrm{K}v = q^3 v, \quad \mathrm{H}_{0,0} v = (q^2 + 1 + q^{-2}) v, \quad \mathrm{H}_{0,1} v = (q^2 - q^{-2}) v.$

$$\begin{tikzcd}
Y_0 Y_2^2 \arrow[dr, dashed, "A_2^{-1}"'] & & & & v \arrow[dr] & & \\
& 2 Y_0 Y_4^{-1} \arrow[d, dashed, "A_0^{-1}"'] \arrow[dr, dashed, "A_2^{-1}"'] & & & & \mathrm{E}^-_{2,0} v, \ \mathrm{E}^-_{2,1} v \arrow[d] \arrow[dr] & \\
& Y_0 Y_4^{-2} \arrow[dr, dashed, "A_2^{-1}"'] & Y_4^{-1} \arrow[d, dashed, "A_0^{-1}"'] & & & \mathrm{E}^-_{0,0} \mathrm{E}^-_{2,0} v \arrow[dr] & \mathrm{E}^-_{2,0} \mathrm{E}^-_{2,1} v \arrow[d] \\
& & Y_2^{-1} Y_4^{-2} & & & & \mathrm{E}^-_{0,0} \mathrm{E}^-_{2,0} \mathrm{E}^-_{2,1} v
\end{tikzcd}$$

Shorthand $\quad \mathrm{E}^-_{k,m} := \mathrm{E}^-_{1,aq^k,m}, \quad \mathrm{H}_{k,m} := \mathrm{H}_{1,aq^k,m}, \quad Y_k := Y_{1,aq^k}, \quad A_k := A_{1,aq^k}.$

$L(Y_0^3 Y_2^2)$: $\mathsf{K}v = q^5 v, \quad \mathsf{H}_{0,0}v = v, \quad \mathsf{H}_{2,0}v = (q^4+q^2+q^{-2}+q^{-4})v, \quad \mathsf{H}_{2,1}v = (q^4+2q^2-2q^{-2}-q^{-4})v.$

$$\begin{tikzcd}
Y_0^3Y_2^2 \arrow[d, "A_0^{-1}"] \arrow[dr, "A_2^{-1}"] & & \\
Y_0^2Y_2 \arrow[dr, "A_2^{-1}"] & 2Y_0^3Y_2Y_4^{-1} \arrow[d, "A_0^{-1}"] \arrow[dr, "A_2^{-1}"] & \\
& 4Y_0^2Y_4^{-1} \arrow[d, "A_0^{-1}"] \arrow[dr, "A_2^{-1}"] & Y_0^3Y_4^{-2} \arrow[d, "A_0^{-1}"] \\
& 2Y_0Y_2^{-1}Y_4^{-1} \arrow[dr, "A_2^{-1}"] & 3Y_0^2Y_2^{-1}Y_4^{-2} \arrow[d, "A_0^{-1}"] \\
& & 3Y_0Y_2^{-2}Y_4^{-2} \arrow[d, "A_0^{-1}"] \\
& & Y_2^{-3}Y_4^{-2}.
\end{tikzcd}$$

$$\begin{tikzcd}
v \arrow[d] \arrow[dr] & & \\
\mathsf{E}^-_{0,0}v \arrow[dr] & \mathsf{E}^-_{2,0}v,\ \mathsf{E}^-_{2,1}v \arrow[d] \arrow[dr] & \\
& \mathsf{E}^-_{0,0}\mathsf{E}^-_{2,0}v,\ \mathsf{E}^-_{0,0}\mathsf{E}^-_{2,1}v,\ \mathsf{E}^-_{0,1}\mathsf{E}^-_{2,0}v,\ \mathsf{E}^-_{0,1}\mathsf{E}^-_{2,1}v \arrow[d] \arrow[dr] & \mathsf{E}^-_{2,0}\mathsf{E}^-_{2,1}v \arrow[d] \\
& \mathsf{E}^-_{0,0}\mathsf{E}^-_{0,1}\mathsf{E}^-_{2,0}v,\ \mathsf{E}^-_{0,0}\mathsf{E}^-_{0,1}\mathsf{E}^-_{2,1}v \arrow[dr] & \mathsf{E}^-_{0,0}\mathsf{E}^-_{2,0}\mathsf{E}^-_{2,1}v,\ \mathsf{E}^-_{0,1}\mathsf{E}^-_{2,0}\mathsf{E}^-_{2,1}v,\ \mathsf{E}^-_{0,2}\mathsf{E} \arrow[d] \\
& & \mathsf{E}^-_{0,0}\mathsf{E}^-_{0,2}\mathsf{E}^-_{2,0}\mathsf{E}^-_{2,1}v,\ \mathsf{E}^-_{0,1}\mathsf{E}^-_{0,2}\mathsf{E}^-_{2,0}\mathsf{E}^-_{2,1}v,\ \mathsf{E}^-_{0,0} \arrow[d] \\
& & \mathsf{E}^-_{0,0}\mathsf{E}^-_{0,1}\mathsf{E}^-_{0,2}\mathsf{E}^-_{2,0}\mathsf{E}^-_{2,1}v.
\end{tikzcd}$$

Outlook

- Basis for $\mathcal{A}$?
  Easy to check that $\mathcal{A} = \mathcal{A}^- . \mathcal{A}^0 . \mathcal{A}^+$.
  Basis for $\mathcal{A}^\pm$? *A canonical basis for $\mathcal{A}^\pm$?*
- Coproduct? From Drinfeld "current" coproduct?
- Rational limit à la Yangians.
- All for $\mathfrak{g}$ semisimple. Does it work for all $\mathfrak{g}$ with symmetrizable Cartan matrix?
  (affinize $U_q(\mathfrak{g})$ with $\mathfrak{g}$ affine $\rightsquigarrow$ one approach to quantum toroidal algebras)
- (in simply-laced types)
  Relation to geometric realizations of representations?