Factorization formulae for $SU(3)$ scalar products

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**SU(2)-invariant models**

- Define $f(\lambda, \mu) = (\lambda - \mu + 1)/(\lambda - \mu)$ and $g(\lambda, \mu) = 1/(\lambda - \mu)$. The $R$-matrix is

$$R_{\alpha\beta}(\lambda, \mu) = \begin{pmatrix}
 f(\lambda, \mu) & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & g(\lambda, \mu) & 1 & 0 \\
 0 & 0 & 0 & f(\lambda, \mu)
\end{pmatrix}_{\alpha\beta}$$

- Represent the entries of the $R$-matrix by

$$[R_{\alpha\beta}(\lambda, \mu)]_{i\alpha j\beta} = \lambda^i_{i\alpha} j^j_{j\beta}$$
Algebraic Bethe Ansatz for $SU(2)$-invariant models

[Faddeev, Sklyanin, Takhtajan ’79], [Izergin, Korepin ’84]

- The monodromy matrix is

$$T_\alpha(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_\alpha$$

and satisfies the intertwining relation

$$R_{\alpha\beta}(\lambda, \mu)T_\alpha(\lambda)T_\beta(\mu) = T_\beta(\mu)T_\alpha(\lambda)R_{\alpha\beta}(\lambda, \mu)$$

- Introduce the pseudo-vacuum states $|0\rangle$, $\langle 0|$ and let the operator entries act according to the rules

  $$A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad C(\lambda)|0\rangle = 0, \quad B(\lambda)|0\rangle \neq 0$$

  $$\langle 0|A(\lambda) = a(\lambda)\langle 0|, \quad \langle 0|D(\lambda) = d(\lambda)\langle 0|, \quad \langle 0|C(\lambda) \neq 0, \quad \langle 0|B(\lambda) = 0$$

where $a(\lambda), d(\lambda)$ are constants.
Algebraic Bethe Ansatz for $SU(2)$-invariant models

- The transfer matrix is $\mathcal{T}(x) = A(x) + D(x)$. Eigenvectors of $\mathcal{T}(x)$ are given by

$$|\lambda_1, \ldots, \lambda_\ell\rangle = B(\lambda_1) \cdots B(\lambda_\ell)|0\rangle$$

$$\langle \lambda_\ell, \ldots, \lambda_1| = \langle 0|C(\lambda_\ell) \cdots C(\lambda_1)$$

with $\{\lambda_1, \ldots, \lambda_\ell\}$ satisfying the Bethe equations

$$r(\lambda_i) \equiv \frac{a(\lambda_i)}{d(\lambda_i)} = -\prod_{j=1}^\ell \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1} \quad \forall \ 1 \leq i \leq \ell$$
Specialization to $SU(2)$-invariant XXX model

- Consider an XXX spin-chain of length $L$. The monodromy matrix becomes

$$T_\alpha(\lambda) = R_{\alpha 1}(\lambda, w_1) \ldots R_{\alpha L}(\lambda, w_L)$$

- The pseudo-vacuum states $|0\rangle$ and $\langle 0|$ are chosen to be

$$|0\rangle = |1^L\rangle \equiv \prod_{i=1}^{L} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_i \quad \quad \langle 0| = \langle 1^L| \equiv \prod_{i=1}^{L} \begin{bmatrix} 1 & 0 \end{bmatrix}_i$$

- For this model, we have $r(\lambda) = \prod_{i=1}^{L} (\lambda - w_i + 1)/(\lambda - w_i)$. 
Definition of scalar product \([\text{Korepin '82}, \text{Izergin, Korepin '84,'85}]\)

- The scalar product \(S(\{\lambda^C\}_\ell | \{\lambda^B\}_\ell)\) is defined as

\[
S(\{\lambda^C\}_\ell | \{\lambda^B\}_\ell) = \frac{\langle \lambda^C_\ell, \ldots, \lambda^C_1 | \lambda^B_1, \ldots, \lambda^B_\ell \rangle}{\prod_{i=1}^\ell d(\lambda^C_i) d(\lambda^B_i)}
\]

- In the model just defined

\[
S(\{\lambda^C\}_\ell | \{\lambda^B\}_\ell) = \frac{1}{\prod_{i=1}^\ell d(\lambda^C_i) d(\lambda^B_i)}
\]
The scalar product can be expressed as $S(\{\lambda^C\}|\{\lambda^B\}) =$

$$
\sum \prod_{\lambda^B} r(\lambda^B_I) \prod_{\lambda^C} r(\lambda^C_{II}) \prod_{\lambda^C,\lambda^C} f(\lambda^C_I, \lambda^C_{II}) \prod_{\lambda^B,\lambda^B} f(\lambda^B_{II}, \lambda^B_I) Z(\lambda^B_{II}|\lambda^C_{II}) Z(\lambda^C_I|\lambda^B_I)
$$

where $\sum$ is over

$$\{\lambda^C\} = \{\lambda^C_I\} \cup \{\lambda^C_{II}\}, \{\lambda^B\} = \{\lambda^B_I\} \cup \{\lambda^B_{II}\}$$

such that $|\lambda^C_I| = |\lambda^B_I|, |\lambda^C_{II}| = |\lambda^B_{II}|$

$$Z(\{\lambda\}_\ell | \{w\}_\ell) =$$

$$\prod_{i,j=1}^\ell (\lambda_i - w_j + 1) \prod_{1 \leq i < j \leq \ell} (\lambda_i - \lambda_j) (w_j - w_i) \times$$

$$\det \left( \frac{1}{(\lambda_i - w_j + 1)(\lambda_i - w_j)} \right)_{1 \leq i, j \leq \ell}$$

[Izergin '87]
Slavnov's determinant expression [Slavnov '89]

- When one set of variables \( \{ \lambda^B \} \) satisfy Bethe equations

\[
S(\{ \lambda^C \}, \{ \lambda^B \}) = \sum (-)^{|\lambda^{B}_I|} \frac{\prod_{j=1}^{\ell} (\frac{\lambda^B_I - \lambda^B_j + 1}{\lambda^B_I - \lambda^B_j - 1}) \prod r(\lambda^C_{II})}{\prod f(\lambda^C_I, \lambda^C_{II}) \prod f(\lambda^B_{II}, \lambda^B_I) Z(\lambda^B_{II} | \lambda^C_{II}) Z(\lambda^C_I | \lambda^B_I)}
\]

- This can be summed to a determinant \( S(\{ \lambda^C \}, \{ \lambda^B \}) = \)

\[
\det \left( \sum_{1 \leq i, j \leq \ell} \frac{1}{\lambda^B_j - \lambda^C_i} \left( \prod_{k \neq j} \left( \frac{\lambda^B_k - \lambda^C_i + 1}{\lambda^B_k - \lambda^C_i - 1} \prod_{k \neq j} (\lambda^B_k - \lambda^C_i) \right) \right) \right)
\]

[Kitanine, Kozlowski, Maillet, Slavnov, Terras '06]
Partial domain wall partition function

- Consider taking the limit

\[ S(\{\lambda^C\}|\{\infty\}) \equiv \frac{1}{\ell!} \lim_{\lambda_\ell^B, \ldots, \lambda_1^B \to \infty} \left( \lambda_\ell^B \cdots \lambda_1^B S(\{\lambda^C\}|\{\lambda^B\}) \right) \]  

(1)

- Performing this limit, we obtain the sum

\[
S(\{\lambda^C\}|\{\infty\}) = \sum_{\{\lambda^C\} = \{\lambda_I^C\} \cup \{\lambda_{II}^C\}} (-)^{|\lambda_I^C|} \prod_{\lambda_I^C} r(\lambda_{II}^C) \prod_{\lambda_I^C, \lambda_{II}^C} f(\lambda_I^C, \lambda_{II}^C)
\]

- In determinant form [Kostov ’12]

\[
S(\{\lambda^C\}|\{\infty\}) = \frac{\det \left( (\lambda_i^C)^{-1} r(\lambda_i^C) - (\lambda_i^C + 1)^{-1} \right)_{1 \leq i, j \leq \ell}}{\prod_{1 \leq i < j \leq \ell} (\lambda_j^C - \lambda_i^C)}
\]
The partial domain wall partition was given its name in [Foda, W ’12].

It comes either as a limiting case of a scalar product,
Partial domain wall partition function

- It comes either as a limiting case of a scalar product,
Partial domain wall partition function

or as a limiting case of a domain wall partition function.

\[ x_{L-\ell} \]

\[ x_1 \]

\[ \lambda_1^C \]

\[ \lambda_{\ell}^C \]

\[ w_1 \]

\[ w_L \]
Partial domain wall partition function

- or as a limiting case of a domain wall partition function.
SU(3)-invariant models

- The $R$-matrix is $R^{(1)}_{\alpha\beta}(\lambda, \mu) =$

\[
\begin{pmatrix}
  f(\lambda, \mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & g(\lambda, \mu) & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & f(\lambda, \mu) & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & g(\lambda, \mu) & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & f(\lambda, \mu) & 0 \\
\end{pmatrix}_{\alpha\beta}
\]

and we define $R^{(2)}_{\alpha\beta}(\lambda, \mu) \equiv R_{\alpha\beta}(\lambda, \mu)$.

- We will also use of the $R$-matrix $R^{*^{(1)}}_{\alpha\beta}(\lambda, \mu) = (R^{(1)}_{\alpha\beta}(-\lambda, -\mu))^t_{\beta}$, given by

\[
\left[R^{*^{(1)}}_{\alpha\beta}(\lambda, \mu)\right]_{i\beta j\beta}^{i\alpha j\alpha} = \lambda_{i\alpha} \rightarrow j_{\alpha}
\]
Nested Bethe Ansatz for $SU(3)$-invariant models

[Kulish, Reshetikhin ’81], [Belliard, Ragoucy ’08]

- The monodromy matrix is

$$T^{(1)}_{\alpha}(\lambda) = \begin{pmatrix} t_{11}(\lambda) & t_{12}(\lambda) & t_{13}(\lambda) \\ t_{21}(\lambda) & t_{22}(\lambda) & t_{23}(\lambda) \\ t_{31}(\lambda) & t_{32}(\lambda) & t_{33}(\lambda) \end{pmatrix} = \begin{pmatrix} A^{(1)}(\lambda) & B^{(1)}(\lambda) \\ C^{(1)}(\lambda) & D^{(1)}(\lambda) \end{pmatrix}_{\alpha}$$

where we have defined

$$B^{(1)}_{\beta}(\lambda) = (t_{12}(\lambda) \quad t_{13}(\lambda))_{\beta} \quad D^{(1)}_{\delta}(\lambda) = \begin{pmatrix} t_{22}(\lambda) \\ t_{32}(\lambda) \end{pmatrix}$$

- The intertwining equation is

$$R^{(1)}_{\alpha\beta}(\lambda - \mu)T^{(1)}_{\alpha}(\lambda)T^{(1)}_{\beta}(\mu) = T^{(1)}_{\beta}(\mu)T^{(1)}_{\alpha}(\lambda)R^{(1)}_{\alpha\beta}(\lambda - \mu)$$
Nested Bethe Ansatz for $SU(3)$-invariant models

- We also need monodromy matrices which are $SU(2)$-invariant, namely

$$T^{(2)}_\alpha (\mu | \lambda, \ldots, \lambda_1) = D^{(1)}_\alpha (\mu) R^{(2)}_{\alpha \lambda} (\mu, \lambda) \cdots R^{(2)}_{\alpha \lambda_1} (\mu, \lambda_1)$$

$$\equiv \left( \begin{array}{ll} A^{(2)} (\mu | \{\lambda\}_\ell) & B^{(2)} (\mu | \{\lambda\}_\ell) \\ C^{(2)} (\mu | \{\lambda\}_\ell) & D^{(2)} (\mu | \{\lambda\}_\ell) \end{array} \right)_{\alpha}$$

and

$$T^{(2)}_\alpha (\lambda, \ldots, \lambda_\ell | \mu) = R^{(2)}_{\alpha \lambda} (\mu, \lambda_1) \cdots R^{(2)}_{\alpha \lambda_\ell} (\mu, \lambda_\ell) D^{(1)}_\alpha (\mu)$$

$$\equiv \left( \begin{array}{ll} A^{(2)} (\{\lambda\}_\ell | \mu) & B^{(2)} (\{\lambda\}_\ell | \mu) \\ C^{(2)} (\{\lambda\}_\ell | \mu) & D^{(2)} (\{\lambda\}_\ell | \mu) \end{array} \right)_{\alpha}$$
Nested Bethe Ansatz for $SU(3)$-invariant models

- The transfer matrix is $T(x) = t_{11}(x) + t_{22}(x) + t_{33}(x)$. Eigenvectors of $T(x)$ are given by

$$|\{\lambda\}_\ell, \{\mu\}_m\rangle = B^{(1)}_{\alpha_1}(\lambda_1) \ldots B^{(1)}_{\alpha_\ell}(\lambda_\ell) B^{(2)}(\mu_1|\{\lambda\}_\ell) \ldots B^{(2)}(\mu_m|\{\lambda\}_\ell)|0\rangle \otimes |2^\ell\rangle_\alpha$$

$$\langle\{\mu\}_m, \{\lambda\}_\ell| = \langle 2^\ell|_\alpha \otimes \langle 0|C^{(2)}(\{\lambda\}_\ell|\mu_m) \ldots C^{(2)}(\{\lambda\}_\ell|\mu_1) C^{(1)}_{\alpha_\ell}(\lambda_\ell) \ldots C^{(1)}_{\alpha_1}(\lambda_1)$$

with $\{\lambda_1, \ldots, \lambda_\ell\}$ and $\{\mu_1, \ldots, \mu_m\}$ satisfying the nested Bethe equations

$$r_1(\lambda_i) \equiv \frac{a_1(\lambda_i)}{a_2(\lambda_i)} = -\prod_{j=1}^\ell \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1} \prod_{k=1}^m f(\mu_k, \lambda_i) \quad \forall \ 1 \leq i \leq \ell$$

$$r_2(\mu_i) \equiv \frac{a_2(\mu_i)}{a_3(\mu_i)} = -\prod_{j=1}^m \frac{\mu_i - \mu_j + 1}{\mu_i - \mu_j - 1} \prod_{k=1}^\ell \frac{1}{f(\mu_i, \lambda_k)} \quad \forall \ 1 \leq i \leq m$$
Specialization to $SU(3)$-invariant XXX model

- Consider an XXX spin-chain of length $L + M$, with the monodromy matrix

$$T^{(1)}_\alpha(\lambda) = R^{(1)}_{\alpha_1}(\lambda, w_1) \cdots R^{(1)}_{\alpha_L}(\lambda, w_L) R^{* (1)}_{\alpha_1'}(\lambda, v_1) \cdots R^{* (1)}_{\alpha_M'}(\lambda, v_M)$$

- The pseudo-vacuum states $|0\rangle$ and $\langle 0|$ are chosen to be

$$|0\rangle = |1^L, 3^M\rangle \equiv \prod_{i=1}^{L} \begin{bmatrix} 1 & \; & \; \\ 0 & 0 & \; \\ \; & 0 & 1 \end{bmatrix}_i \otimes \prod_{j=1}^{M} \begin{bmatrix} 0 & \; & \; \\ 0 & 0 & \; \\ \; & 1 & \; \end{bmatrix}_j$$

$$\langle 0| = \langle 1^L, 3^M| \equiv \prod_{i=1}^{L} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}_i \otimes \prod_{j=1}^{M} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}_j$$

- For this model, we have

$$r_1(\lambda) = \prod_{i=1}^{L} (\lambda - w_i + 1)/(\lambda - w_i) \quad r_2(\mu) = \prod_{j=1}^{M} (v_j - \mu)/(v_j - \mu + 1)$$
The scalar product is defined as:

\[ S(\{\mu^C\}_m, \{\lambda^C\}_\ell|\{\lambda^B\}_\ell, \{\mu^B\}_m) = \]

\[ \prod_{i=1}^{m} \prod_{j=1}^{\ell} f(\mu_i^C, \lambda_j^C) f(\mu_i^B, \lambda_j^B) \frac{\langle \{\mu^C\}_m, \{\lambda^C\}_\ell|\{\lambda^B\}_\ell, \{\mu^B\}_m \rangle}{\prod_{i=1}^{\ell} a_2(\lambda_i^C) a_2(\lambda_i^B) \prod_{j=1}^{m} a_3(\mu_j^C) a_3(\mu_j^B)} \]

Evaluated as a sum, for generic values of the parameters, in [Reshetikhin '86].

When \(\{\lambda^B\} = \{\lambda^C\}\) and \(\{\mu^B\} = \{\mu^C\}\), and \(\{\lambda^B\}, \{\mu^B\}\) satisfy the Bethe equations, it is known in determinant form, [Reshetikhin '86].

More recently, this result was generalized to allow \(\{\lambda^B\} \neq \{\lambda^C\}\) and \(\{\mu^B\} \neq \{\mu^C\}\), if \(\{\lambda^C\}, \{\mu^C\}\) parametrize a twisted Bethe eigenstate, [Belliard, Pakuliak, Ragoucy, Slavnov '12].

We are interested in the case where just \(\{\lambda^B\}, \{\mu^B\}\) satisfy Bethe equations, while \(\{\lambda^C\}, \{\mu^C\}\) are free.
Graphical representation (1)
Graphical representation (2)
Sum expression for $SU(3)$ scalar product \cite{Reshetikhin '86}

- To save space, let

$$f(\mu, \lambda) = \prod_{i=1}^{m} \prod_{j=1}^{\ell} f(\mu_i, \lambda_j)$$

for $\mu = \{\mu_1, \ldots, \mu_m\}, \lambda = \{\lambda_1, \ldots, \lambda_\ell\}$.

- The scalar product can be expressed as

$$S(\{\mu^C\}, \{\lambda^C\}|\{\lambda^B\}, \{\mu^B\}) =$$

$$\sum \prod r_1(\lambda^B_1) \prod r_1(\lambda^C_\Pi) \prod r_2(\mu^B_\Pi) \prod r_2(\mu^C_\Pi) f(\lambda^C_1, \lambda^C_\Pi) f(\lambda^B_1, \lambda^B_\Pi) f(\mu^C_1, \mu^C_\Pi) f(\mu^B_1, \mu^B_\Pi)$$

$$\times f(\mu^C_1, \lambda^B_1) f(\mu^B_1, \lambda^B_\Pi) Z(\{\lambda^B_1\}, \{\mu^B_1\}|\{\lambda^C_1\}, \{\mu^C_1\}) Z(\{\lambda^C_\Pi\}, \{\mu^C_\Pi\}|\{\lambda^B_\Pi\}, \{\mu^B_\Pi\})$$

where $\sum$ is over

- $\{\lambda^C\} = \{\lambda^C_1\} \cup \{\lambda^C_\Pi\}$, $\{\lambda^B\} = \{\lambda^B_1\} \cup \{\lambda^B_\Pi\}$ such that $|\lambda^C_1| = |\lambda^B_1|, |\lambda^C_\Pi| = |\lambda^B_\Pi|$

- $\{\mu^C\} = \{\mu^C_1\} \cup \{\mu^C_\Pi\}$, $\{\mu^B\} = \{\mu^B_1\} \cup \{\mu^B_\Pi\}$ such that $|\mu^C_1| = |\mu^B_1|, |\mu^C_\Pi| = |\mu^B_\Pi|$
Reshetikhin’s partition function  [Reshetikhin ’86]

- The quantity introduced in the sum above, $Z(\{\lambda\}, \{\mu\} | \{w\}, \{v\})$, is given by

- It is a candidate for the ‘domain wall partition function’ in the $SU(3)$ XXX spin-chain.
Reshetikhin’s partition function

- It can be expressed as

\[
Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\}) = \sum \prod_{\mu_I, \mu_{II}} f(\mu_I, \mu_{II}) \prod_{\lambda_I, \lambda_{II}} f(\lambda_{II}, \lambda_I) \prod_{\mu_I, \lambda_I} f(\mu_I, \lambda_I) \\
\times Z(\{\lambda_{II}\}|\{\mu_{II}\})Z(\{\lambda_I\} \cup \{\mu_{II}\}|\{w\})Z(\{v\}|\{\mu_I\} \cup \{\lambda_{II}\})
\]

where \(\sum\) is over

\[
\{\lambda\} = \{\lambda_I\} \cup \{\lambda_{II}\}, \{\mu\} = \{\mu_I\} \cup \{\mu_{II}\} \text{ such that } |\lambda_{II}| = |\mu_{II}|
\]

[W ’12], [Belliard, Pakuliak, Ragoucy, Slavnov ’12]

- Some limiting cases

\[
Z(\{\lambda\}_\ell, \{\infty\}_m|\{w\}_\ell, \{v\}_m) = (-)^m Z(\{\lambda\}|\{w\})
\]

\[
Z(\{\infty\}_\ell, \{\mu\}_m|\{w\}_\ell, \{v\}_m) = Z(\{v\}|\{\mu\})
\]

\[
Z(\{\lambda\}_\ell, \{\mu\}_m|\{w\}_\ell, \{\infty\}_m) = \prod_{i=1}^m \prod_{j=1}^\ell f(\mu_i, w_j)Z(\{\lambda\}|\{w\})
\]

\[
Z(\{\lambda\}_\ell, \{\mu\}_m|\{\infty\}_\ell, \{v\}_m) = (-)^\ell \prod_{i=1}^m \prod_{j=1}^\ell f(v_i, \lambda_j)Z(\{v\}|\{\mu\})
\]
Introducing the Bethe equations

- Returning to the scalar product, if \( \{ \lambda^B \}, \{ \mu^B \} \) satisfy the Bethe equations, then

\[
S(\{ \mu^C \}, \{ \lambda^C \} | \{ \lambda^B \}, \{ \mu^B \}) = 
\sum (-)^{|\lambda^B_I| + |\mu^B_I|} \prod_{\lambda^B_I} \left( \prod_{j=1}^{\ell} \left( \frac{\lambda^B_I - \lambda^B_j + 1}{\lambda^B_I - \lambda^B_j - 1} \right) \prod_{k=1}^{m} f(\mu^B_k, \lambda^B_I) \right) \times 
\prod_{\mu^B_I} \left( \prod_{j=1}^{m} \left( \frac{\mu^B_I - \mu^B_j + 1}{\mu^B_I - \mu^B_j - 1} \right) \prod_{k=1}^{\ell} \frac{1}{f(\mu^B_I, \lambda^B_k)} \right) \prod_{\lambda^C_I} r_1(\lambda^C_I) \prod_{\mu^C_I} r_2(\mu^C_I) \times 
f(\lambda^C_I, \lambda^C_I) f(\lambda^B_I, \lambda^B_I) f(\mu^C_I, \mu^C_I) f(\mu^B_I, \mu^B_I) f(\mu^C_I, \lambda^B_I) f(\mu^B_I, \lambda^B_I) \times 
Z(\{ \lambda^B_I \}, \{ \mu^C_I \} | \{ \lambda^C_I \}, \{ \mu^B_I \}) Z(\{ \lambda^C_I \}, \{ \mu^B_I \} | \{ \lambda^B_I \}, \{ \mu^C_I \})
\]

- The goal is to sum this to a simpler expression, as was possible in the \( SU(2) \) case.

- However, because the partition function \( Z \) is not known as a determinant, the way to do this is not obvious.
Taking the limit $\mu_1^B, \ldots, \mu_m^B \to \infty$
Taking the limit $\mu_1^B, \ldots, \mu_m^B \to \infty$

- Evaluating this limit directly from the sum form, we obtain

$$S(\{\mu^C_i\}, \{\lambda^C\}|\{\lambda^B\}, \{\infty\}) = \frac{1}{m!} \sum_{\{\lambda^C\} = \{\lambda^C_I\} \cup \{\lambda^C_{II}\}} \sum_{\{\lambda^B\} = \{\lambda^B_I\} \cup \{\lambda^B_{II}\}} (-)^{\lambda^B} \binom{m}{k}$$

$$\times \prod_1 r_1(\lambda^C_{II}) \prod_{\mu^C_1} \prod_{\lambda^B_j} \prod_{\mu^C} \prod_{\lambda^B_j} \prod_{\ell} \left( \frac{\lambda^B - \lambda^B_j + 1}{\lambda^B - \lambda^B_j - 1} \right) f(\lambda^C_I, \lambda^C_{II}) f(\lambda^B_{II}, \lambda^B_I) f(\mu^C_I, \mu^C_{II}) f(\mu^C_I, \lambda^C_I)$$

$$\times (m - k)! f(\mu^C_I, \lambda^C_{II}) Z(\{\lambda^B_{II}\}|\{\lambda^C_{II}\})(-)^k k! Z(\{\lambda^C_I\}|\{\lambda^B_I\}) \right)$$

- In fact the above sum factorizes as

$$\left( \sum (-)^{\mu^C_{II}} \prod_{\mu^C_1} \left( r_2(\mu^C_I) \prod_{k=1}^\ell f(\mu^C_I, \lambda^C_k) \right) f(\mu^C_{II}, \mu^C_I) \right) \left( \sum (-)^{\lambda^B} \right)$$

$$\prod_{\lambda^B_j} \prod_1 \left( \frac{\lambda^B - \lambda^B_j + 1}{\lambda^B - \lambda^B_j - 1} \right) \prod_{\lambda^C_{II}} r_1(\lambda^C_{II}) f(\lambda^C_I, \lambda^C_{II}) f(\lambda^B_{II}, \lambda^B_I) Z(\{\lambda^B_{II}\}|\{\lambda^C_{II}\}) Z(\{\lambda^C_I\}|\{\lambda^B_I\}) \right)$$
Result of the limit $\mu_1^B, \ldots, \mu_m^B \to \infty$

- Both of the factors are sums which we know how to evaluate. They are a partial domain wall partition function, and a Slavnov scalar product, from $SU(2)$ theory. Hence

$$S(\{\mu^C\}, \{\lambda^C\}||\{\lambda^B\}, \{\infty\}) =$$

$$\det \left( (\mu_i^C)^{j-1} r_2(\mu_i^C) \prod_{k=1}^\ell \left( \frac{\mu_i^C - \lambda_k^C + 1}{\mu_i^C - \lambda_k^C} \right) - (\mu_i^C + 1)^{j-1} \right)_{1 \leq i, j \leq m} \times$$

$$\prod_{1 \leq i < j \leq m} (\mu_j^C - \mu_i^C)$$

$$\det \left( \frac{1}{\lambda_j^B - \lambda_i^C} \left( \prod_{k \neq j} (\lambda_k^B - \lambda_i^C + 1) r_1(\lambda_i^C) - \prod_{k \neq j} (\lambda_k^B - \lambda_i^C - 1) \right) \right)_{1 \leq i, j \leq \ell}$$

$$\prod_{1 \leq i < j \leq \ell} (\lambda_j^C - \lambda_i^C)(\lambda_i^B - \lambda_j^B)$$

[W '12]
Taking the limit $\lambda_1^B, \ldots, \lambda_\ell^B \to \infty$
Taking the limit $\lambda^B_1, \ldots, \lambda^B_\ell \to \infty$

- Evaluating this limit directly from the sum form, we obtain

$$S(\{\mu^C\}, \{\lambda^C\}|\{\infty\}, \{\mu^B\}) = \frac{1}{\ell!} \sum_{\{\mu^C\} = \{\mu^C_1\} \cup \{\mu^C_\Pi\}} \sum_{\{\mu^B\} = \{\mu^B_1\} \cup \{\mu^B_\Pi\}} (-)^{|\mu^B_\Pi|} \binom{\ell}{k}$$

$$\times \prod_{\lambda^C_\Pi} r_1(\lambda^C_\Pi) \prod_{\mu^C_1} r_2(\mu^C_1) \prod_{\mu^B_1} \prod_{j=1}^m \left( \frac{\mu^B_\Pi - \mu^B_j + 1}{\mu^B_\Pi - \mu^B_j - 1} \right) f(\lambda^C_1, \lambda^C_1) f(\mu^C_1, \mu^C_1) f(\mu^B_1, \mu^B_\Pi) f(\mu^C_1, \lambda^C_1)$$

$$\times k! Z(\{\mu^B_1\}|\{\mu^C_1\}) (-)^{\ell-k} (\ell-k)! f(\mu^C_1, \lambda^C_1) Z(\{\mu^C_1\}|\{\mu^B_1\})$$

- Once again, the above sum has a clear factorization

$$f(\mu^C, \lambda^C) \left( \sum (-)^{|\lambda^C_\Pi|} \prod_{\lambda^C_\Pi} \left( r_1(\lambda^C_\Pi) \prod_{k=1}^m \frac{1}{f(\mu^C_k, \lambda^C_\Pi)} \right) f(\lambda^C_1, \lambda^C_1) \right) \left( \sum (-)^{|\mu^B_\Pi|} \prod_{\mu^B_1} \prod_{j=1}^m \left( \frac{\mu^B_\Pi - \mu^B_j + 1}{\mu^B_\Pi - \mu^B_j - 1} \right) \prod_{\mu^C_1} r_2(\mu^C_1) f(\mu^C_1, \mu^C_1) f(\mu^B_1, \mu^B_\Pi) Z(\{\mu^B_1\}|\{\mu^C_1\}) Z(\{\mu^C_1\}|\{\mu^B_1\}) \right)$$
Result of the limit $\lambda_1^B, \ldots, \lambda_\ell^B \to \infty$

- As before, both sums are known from $SU(2)$ results. Hence

$$S(\{\mu^C\}, \{\lambda^C\}|\{\infty\}, \{\mu^B\}) =$$

$$\frac{\det \left( (\lambda_i^C)^{j-1} r_1(\lambda_i^C) - (\lambda_i^C + 1)^{j-1} \prod_{k=1}^m \left( \frac{\mu_k^C - \lambda_i^C + 1}{\mu_k^C - \lambda_i^C} \right) \right)_{1 \leq i,j \leq \ell}}{\prod_{1 \leq i < j \leq \ell} (\lambda_j^C - \lambda_i^C)} \times$$

$$\frac{\det \left( \frac{1}{\mu_j^B - \mu_i^C} \left( \prod_{k \neq j} (\mu_k^B - \mu_i^C + 1) r_2(\mu_i^C) - \prod_{k \neq j} (\mu_k^B - \mu_i^C - 1) \right) \right)_{1 \leq i,j \leq m}}{\prod_{1 \leq i < j \leq m} (\mu_j^C - \mu_i^C)(\mu_i^B - \mu_j^B)}$$

[W ’12]
Future work

- Can we use the two limits calculated here, as well as the work of [Reshetikhin ’86], [Belliard, Pakuliak, Ragoucy, Slavnov ’12], to find a manageable expression for the Bethe scalar product?

- Recently, it was shown in [Kostov, Matsuo ’12], [Foda, W ’12] that the XXX $SU(2)$ Bethe scalar product is in fact a partial domain wall partition function.

- Can a similar statement be made with respect to the XXX $SU(3)$ Bethe scalar product, and a ‘partial’ version of Reshetikhin’s partition function?

- Generalization to $SU(n)$ models. The norm has already been conjectured in [Escobedo, Gromov, Sever, Vieira ’11].