Factorization formulae for $SU(3)$ scalar products

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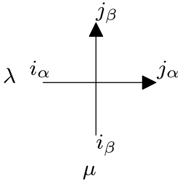
$SU(2)$-invariant models

- Define $f(\lambda,\mu) = (\lambda-\mu+1)/(\lambda-\mu)$ and $g(\lambda,\mu) = 1/(\lambda-\mu)$. The R-matrix is

$$R_{\alpha\beta}(\lambda,\mu) = \left(\begin{array}{cc|cc} f(\lambda,\mu) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda,\mu) & 0 \\ \hline 0 & g(\lambda,\mu) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda,\mu) \end{array}\right)_{\alpha\beta}$$

- Represent the entries of the R-matrix by

$$\Big[R_{\alpha\beta}(\lambda,\mu)\Big]^{i_\alpha j_\alpha}_{i_\beta j_\beta} =$$



Algebraic Bethe Ansatz for $SU(2)$-invariant models

[Faddeev, Sklyanin, Takhtajan '79], [Izergin, Korepin '84]

- The monodromy matrix is

$$T_{\alpha}(\lambda) = \left(\begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right)_{\alpha}$$

  and satisfies the intertwining relation

$$R_{\alpha\beta}(\lambda,\mu) T_{\alpha}(\lambda) T_{\beta}(\mu) = T_{\beta}(\mu) T_{\alpha}(\lambda) R_{\alpha\beta}(\lambda,\mu)$$

- Introduce the *pseudo-vacuum* states $|0\rangle, \langle 0|$ and let the operator entries act according to the rules

$$\begin{array}{llll} A(\lambda)|0\rangle = a(\lambda)|0\rangle, & D(\lambda)|0\rangle = d(\lambda)|0\rangle, & C(\lambda)|0\rangle = 0, & B(\lambda)|0\rangle \neq 0 \\ \langle 0|A(\lambda) = a(\lambda)\langle 0|, & \langle 0|D(\lambda) = d(\lambda)\langle 0|, & \langle 0|C(\lambda) \neq 0, & \langle 0|B(\lambda) = 0 \end{array}$$

  where $a(\lambda), d(\lambda)$ are constants.

Algebraic Bethe Ansatz for $SU(2)$-invariant models

- The transfer matrix is $\mathcal{T}(x) = A(x) + D(x)$. Eigenvectors of $\mathcal{T}(x)$ are given by

$$|\lambda_1, \ldots, \lambda_\ell\rangle = B(\lambda_1) \ldots B(\lambda_\ell)|0\rangle$$

$$\langle\lambda_\ell, \ldots, \lambda_1| = \langle 0|C(\lambda_\ell) \ldots C(\lambda_1)$$

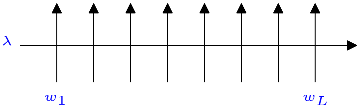
with $\{\lambda_1, \ldots, \lambda_\ell\}$ satisfying the Bethe equations

$$r(\lambda_i) \equiv \frac{a(\lambda_i)}{d(\lambda_i)} = -\prod_{j=1}^{\ell} \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1} \qquad \forall\ 1 \leq i \leq \ell$$

Specialization to $SU(2)$-invariant XXX model

- Consider an XXX spin-chain of length L. The monodromy matrix becomes

$$T_\alpha(\lambda) = R_{\alpha 1}(\lambda, w_1) \ldots R_{\alpha L}(\lambda, w_L)$$



- The pseudo-vacuum states $|0\rangle$ and $\langle 0|$ are chosen to be

$$|0\rangle = |1^L\rangle \equiv \prod_{i=1}^{L} \left[\begin{array}{c} 1 \\ 0 \end{array} \right]_i \qquad \langle 0| = \langle 1^L| \equiv \prod_{i=1}^{L} \left[\begin{array}{cc} 1 & 0 \end{array} \right]_i$$

- For this model, we have $r(\lambda) = \prod_{i=1}^{L} (\lambda - w_i + 1)/(\lambda - w_i)$.

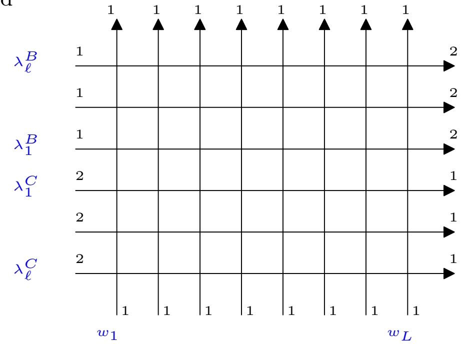
Definition of scalar product [Korepin '82], [Izergin, Korepin '84,'85]

- The scalar product $S(\{\lambda^C\}_\ell|\{\lambda^B\}_\ell)$ is defined as

$$S(\{\lambda^C\}_\ell|\{\lambda^B\}_\ell) = \frac{\langle \lambda^C_\ell, \ldots, \lambda^C_1 | \lambda^B_1, \ldots, \lambda^B_\ell \rangle}{\prod_{i=1}^{\ell} d(\lambda^C_i) d(\lambda^B_i)}$$

- In the model just defined

$$S(\{\lambda^C\}_\ell|\{\lambda^B\}_\ell) =$$



Sum expression for $SU(2)$ scalar product [Izergin, Korepin '84]

- The scalar product can be expressed as $S(\{\lambda^C\}|\{\lambda^B\}) =$

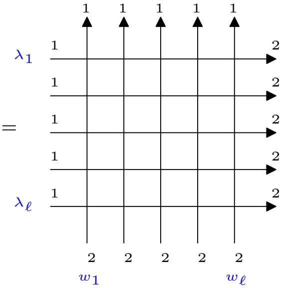
$$\sum \prod_{\lambda_{\rm I}^B} r(\lambda_{\rm I}^B) \prod_{\lambda_{\rm II}^C} r(\lambda_{\rm II}^C) \prod_{\lambda_{\rm I}^C,\lambda_{\rm II}^C} f(\lambda_{\rm I}^C,\lambda_{\rm II}^C) \prod_{\lambda_{\rm I}^B,\lambda_{\rm II}^B} f(\lambda_{\rm II}^B,\lambda_{\rm I}^B) Z(\lambda_{\rm II}^B|\lambda_{\rm II}^C) Z(\lambda_{\rm I}^C|\lambda_{\rm I}^B)$$

where $\sum$ is over

$$\{\lambda^C\} = \{\lambda_{\rm I}^C\} \cup \{\lambda_{\rm II}^C\}, \{\lambda^B\} = \{\lambda_{\rm I}^B\} \cup \{\lambda_{\rm II}^B\} \text{ such that } |\lambda_{\rm I}^C| = |\lambda_{\rm I}^B|, |\lambda_{\rm II}^C| = |\lambda_{\rm II}^B|$$

$$Z(\{\lambda\}_\ell|\{w\}_\ell) =$$

[Korepin '82]



$$= \frac{\prod_{i,j=1}^{\ell}(\lambda_i - w_j + 1)}{\prod_{1\le i<j\le \ell}(\lambda_i - \lambda_j)(w_j - w_i)} \times \det\left(\frac{1}{(\lambda_i - w_j + 1)(\lambda_i - w_j)}\right)_{1\le i,j\le \ell}$$

[Izergin '87]

Slavnov's determinant expression [Slavnov '89]

- When one set of variables $\{\lambda^B\}$ satisfy Bethe equations

$$S(\{\lambda^C\},\{\lambda^B\}) = \sum (-)^{|\lambda^B_{\mathrm{I}}|} \prod_{\lambda^B_{\mathrm{I}}} \prod_{j=1}^{\ell} \left(\frac{\lambda^B_{\mathrm{I}} - \lambda^B_j + 1}{\lambda^B_{\mathrm{I}} - \lambda^B_j - 1} \right) \prod_{\lambda^C_{\mathrm{II}}} r(\lambda^C_{\mathrm{II}})$$

$$\times \prod_{\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}} f(\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}) \prod_{\lambda^B_{\mathrm{I}},\lambda^B_{\mathrm{II}}} f(\lambda^B_{\mathrm{II}},\lambda^B_{\mathrm{I}}) Z(\lambda^B_{\mathrm{II}}|\lambda^C_{\mathrm{II}}) Z(\lambda^C_{\mathrm{I}}|\lambda^B_{\mathrm{I}})$$

- This can be summed to a determinant $S(\{\lambda^C\},\{\lambda^B\}) =$

$$\frac{\det\left(\frac{1}{\lambda^B_j - \lambda^C_i} \left(\prod_{k\neq j}^{\ell} (\lambda^B_k - \lambda^C_i + 1) r(\lambda^C_i) - \prod_{k\neq j}^{\ell} (\lambda^B_k - \lambda^C_i - 1) \right) \right)_{1\leq i,j\leq \ell}}{\prod_{1\leq i<j\leq \ell} (\lambda^C_j - \lambda^C_i)(\lambda^B_i - \lambda^B_j)}$$

[Kitanine, Kozlowski, Maillet, Slavnov, Terras '06]

Partial domain wall partition function

- Consider taking the limit

$$S(\{\lambda^C\}|\{\infty\}) \equiv \frac{1}{\ell!} \lim_{\lambda_\ell^B,\ldots,\lambda_1^B \to \infty} \left(\lambda_\ell^B \cdots \lambda_1^B S(\{\lambda^C\}|\{\lambda^B\}) \right) \tag{1}$$

- Performing this limit, we obtain the sum

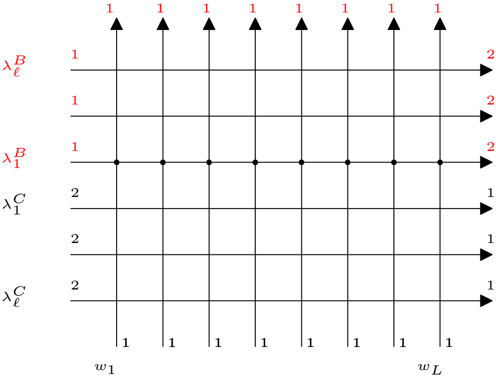
$$S(\{\lambda^C\}|\{\infty\}) = \sum_{\{\lambda^C\}=\{\lambda^C_{\mathrm{I}}\}\cup\{\lambda^C_{\mathrm{II}}\}} (-)^{|\lambda^C_{\mathrm{I}}|} \prod_{\lambda^C_{\mathrm{II}}} r(\lambda^C_{\mathrm{II}}) \prod_{\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}} f(\lambda^C_{\mathrm{I}}, \lambda^C_{\mathrm{II}})$$

- In determinant form [Kostov '12]

$$S(\{\lambda^C\}|\{\infty\}) = \frac{\det\left((\lambda_i^C)^{j-1} r(\lambda_i^C) - (\lambda_i^C + 1)^{j-1} \right)_{1\leq i,j\leq \ell}}{\prod\limits_{1\leq i<j\leq \ell} (\lambda_j^C - \lambda_i^C)}$$

Partial domain wall partition function

- The partial domain wall partition was given its name in [Foda, W '12].
- It comes either as a limiting case of a scalar product,



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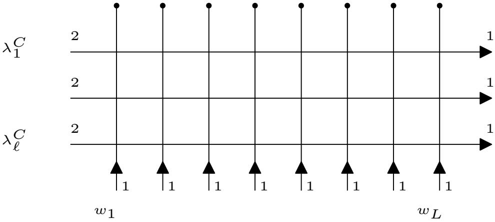

Partial domain wall partition function

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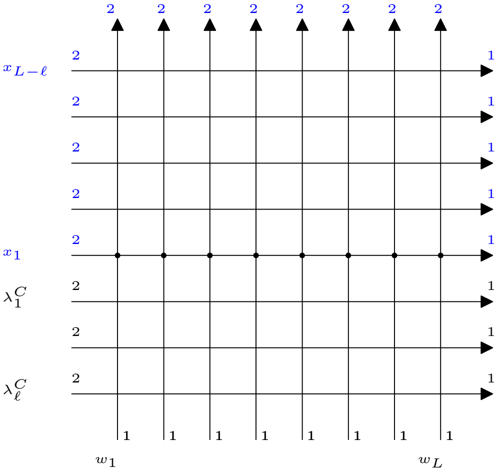

Partial domain wall partition function

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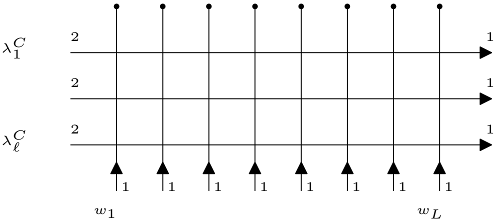


$SU(3)$-invariant models

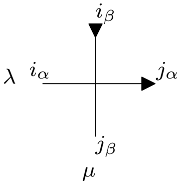
- The R-matrix is $R^{(1)}_{\alpha\beta}(\lambda,\mu) =$

$$\left(\begin{array}{ccc|ccc|ccc}
f(\lambda,\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & g(\lambda,\mu) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & g(\lambda,\mu) & 0 & 0 \\
\hline
0 & g(\lambda,\mu) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f(\lambda,\mu) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & g(\lambda,\mu) & 0 \\
\hline
0 & 0 & g(\lambda,\mu) & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g(\lambda,\mu) & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f(\lambda,\mu)
\end{array}\right)_{\alpha\beta}$$

and we define $R^{(2)}_{\alpha\beta}(\lambda,\mu) \equiv R_{\alpha\beta}(\lambda,\mu)$.

- We will also use of the R-matrix $R^{*(1)}_{\alpha\beta}(\lambda,\mu) = (R^{(1)}_{\alpha\beta}(-\lambda,-\mu))^{t_\beta}$, given by

$$\left[R^{*(1)}_{\alpha\beta}(\lambda,\mu)\right]^{i_\alpha j_\alpha}_{i_\beta j_\beta} =$$



Nested Bethe Ansatz for $SU(3)$-invariant models

[Kulish, Reshetikhin '81], [Belliard, Ragoucy '08]

- The monodromy matrix is

$$T^{(1)}_{\alpha}(\lambda) = \left(\begin{array}{ccc} t_{11}(\lambda) & t_{12}(\lambda) & t_{13}(\lambda) \\ t_{21}(\lambda) & t_{22}(\lambda) & t_{23}(\lambda) \\ t_{31}(\lambda) & t_{32}(\lambda) & t_{33}(\lambda) \end{array} \right)_{\alpha} = \left(\begin{array}{cc} A^{(1)}(\lambda) & B^{(1)}_{\beta}(\lambda) \\ C^{(1)}_{\gamma}(\lambda) & D^{(1)}_{\delta}(\lambda) \end{array} \right)$$

  where we have defined

$$B^{(1)}_{\beta}(\lambda) = \left(\begin{array}{cc} t_{12}(\lambda) & t_{13}(\lambda) \end{array} \right)_{\beta} \qquad D^{(1)}_{\delta}(\lambda) = \left(\begin{array}{cc} t_{22}(\lambda) & t_{23}(\lambda) \\ t_{32}(\lambda) & t_{33}(\lambda) \end{array} \right)_{\delta}$$

- The intertwining equation is

$$R^{(1)}_{\alpha\beta}(\lambda-\mu) T^{(1)}_{\alpha}(\lambda) T^{(1)}_{\beta}(\mu) = T^{(1)}_{\beta}(\mu) T^{(1)}_{\alpha}(\lambda) R^{(1)}_{\alpha\beta}(\lambda-\mu)$$

Nested Bethe Ansatz for $SU(3)$-invariant models

- We also need monodromy matrices which are $SU(2)$-invariant, namely

$$T^{(2)}_{\alpha}(\mu|\lambda_\ell,\ldots,\lambda_1) = D^{(1)}_{\alpha}(\mu) R^{(2)}_{\alpha\alpha_\ell}(\mu,\lambda_\ell)\ldots R^{(2)}_{\alpha\alpha_1}(\mu,\lambda_1)$$

$$\equiv \begin{pmatrix} A^{(2)}(\mu|\{\lambda\}_\ell) & B^{(2)}(\mu|\{\lambda\}_\ell) \\ C^{(2)}(\mu|\{\lambda\}_\ell) & D^{(2)}(\mu|\{\lambda\}_\ell) \end{pmatrix}_{\alpha}$$

and

$$T^{(2)}_{\alpha}(\lambda_1,\ldots,\lambda_\ell|\mu) = R^{(2)}_{\alpha\alpha_1}(\mu,\lambda_1)\ldots R^{(2)}_{\alpha\alpha_\ell}(\mu,\lambda_\ell) D^{(1)}_{\alpha}(\mu)$$

$$\equiv \begin{pmatrix} A^{(2)}(\{\lambda\}_\ell|\mu) & B^{(2)}(\{\lambda\}_\ell|\mu) \\ C^{(2)}(\{\lambda\}_\ell|\mu) & D^{(2)}(\{\lambda\}_\ell|\mu) \end{pmatrix}_{\alpha}$$

Nested Bethe Ansatz for $SU(3)$-invariant models

- The transfer matrix is $\mathcal{T}(x) = t_{11}(x) + t_{22}(x) + t_{33}(x)$. Eigenvectors of $\mathcal{T}(x)$ are given by

$$|\{\lambda\}_\ell, \{\mu\}_m\rangle = B^{(1)}_{\alpha_1}(\lambda_1)\ldots B^{(1)}_{\alpha_\ell}(\lambda_\ell) B^{(2)}(\mu_1|\{\lambda\}_\ell)\ldots B^{(2)}(\mu_m|\{\lambda\}_\ell)|0\rangle \otimes |2^\ell\rangle_\alpha$$

$$\langle\{\mu\}_m, \{\lambda\}_\ell| = \langle 2^\ell|_\alpha \otimes \langle 0|C^{(2)}(\{\lambda\}_\ell|\mu_m)\ldots C^{(2)}(\{\lambda\}_\ell|\mu_1) C^{(1)}_{\alpha_\ell}(\lambda_\ell)\ldots C^{(1)}_{\alpha_1}(\lambda_1)$$

with $\{\lambda_1, \ldots, \lambda_\ell\}$ and $\{\mu_1, \ldots, \mu_m\}$ satisfying the nested Bethe equations

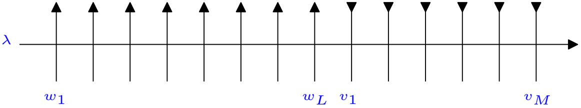
$$r_1(\lambda_i) \equiv \frac{a_1(\lambda_i)}{a_2(\lambda_i)} = -\prod_{j=1}^{\ell} \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1} \prod_{k=1}^{m} f(\mu_k, \lambda_i) \quad \forall\ 1 \le i \le \ell$$

$$r_2(\mu_i) \equiv \frac{a_2(\mu_i)}{a_3(\mu_i)} = -\prod_{j=1}^{m} \frac{\mu_i - \mu_j + 1}{\mu_i - \mu_j - 1} \prod_{k=1}^{\ell} \frac{1}{f(\mu_i, \lambda_k)} \quad \forall\ 1 \le i \le m$$

Specialization to $SU(3)$-invariant XXX model

- Consider an XXX spin-chain of length $L+M$, with the monodromy matrix

$$T^{(1)}_{\alpha}(\lambda) = R^{(1)}_{\alpha 1}(\lambda, w_1) \ldots R^{(1)}_{\alpha L}(\lambda, w_L) R^{*(1)}_{\alpha 1'}(\lambda, v_1) \ldots R^{*(1)}_{\alpha M'}(\lambda, v_M)$$



- The pseudo-vacuum states $|0\rangle$ and $\langle 0|$ are chosen to be

$$|0\rangle = |1^L, 3^M\rangle \equiv \prod_{i=1}^{L} \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]_i \otimes \prod_{j=1}^{M} \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]_j$$

$$\langle 0| = \langle 1^L, 3^M| \equiv \prod_{i=1}^{L} \left[\begin{array}{ccc} 1 & 0 & 0 \end{array} \right]_i \otimes \prod_{j=1}^{M} \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right]_j$$

- For this model, we have

$$r_1(\lambda) = \prod_{i=1}^{L} (\lambda - w_i + 1)/(\lambda - w_i) \quad r_2(\mu) = \prod_{j=1}^{M} (v_j - \mu)/(v_j - \mu + 1)$$

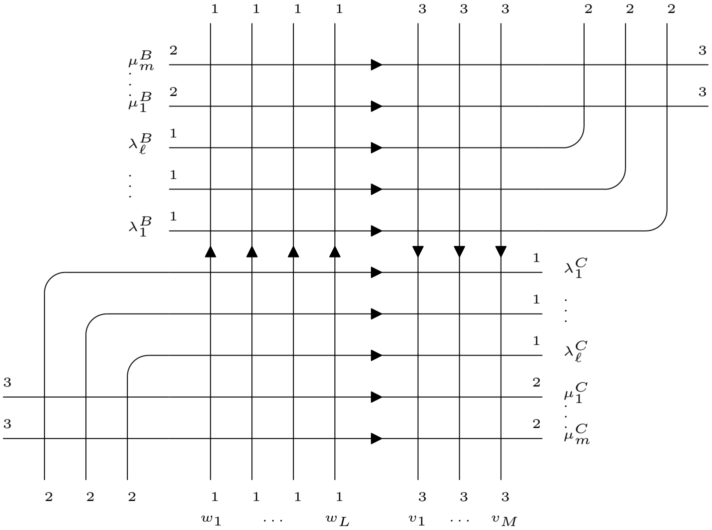
Definition of scalar product [Reshetikhin '86]

- The scalar product is defined as $S(\{\mu^C\}_m, \{\lambda^C\}_\ell | \{\lambda^B\}_\ell, \{\mu^B\}_m) =$

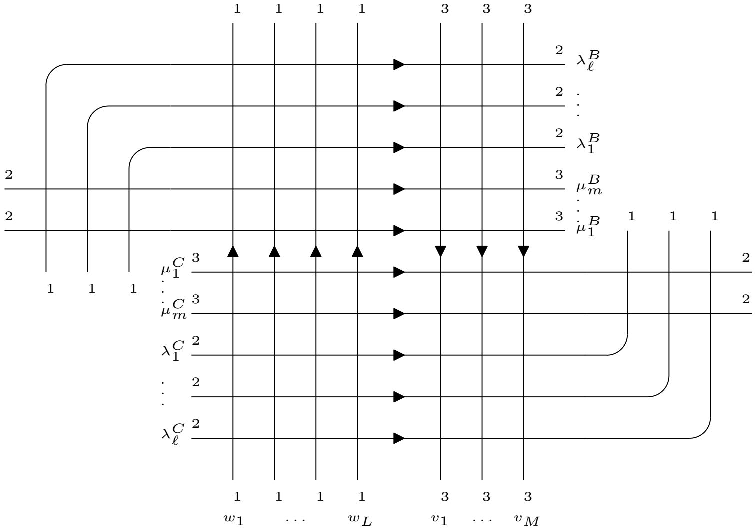
$$\prod_{i=1}^{m} \prod_{j=1}^{\ell} f(\mu_i^C, \lambda_j^C) f(\mu_i^B, \lambda_j^B) \frac{\langle \{\mu^C\}_m, \{\lambda^C\}_\ell | \{\lambda^B\}_\ell, \{\mu^B\}_m \rangle}{\prod_{i=1}^{\ell} a_2(\lambda_i^C) a_2(\lambda_i^B) \prod_{j=1}^{m} a_3(\mu_j^C) a_3(\mu_j^B)}$$

- Evaluated as a sum, for generic values of the parameters, in [Reshetikhin '86].
- When $\{\lambda^B\} = \{\lambda^C\}$ and $\{\mu^B\} = \{\mu^C\}$, and $\{\lambda^B\}, \{\mu^B\}$ satisfy the Bethe equations, it is known in determinant form, [Reshetikhin '86].
- More recently, this result was generalized to allow $\{\lambda^B\} \neq \{\lambda^C\}$ and $\{\mu^B\} \neq \{\mu^C\}$, if $\{\lambda^C\}, \{\mu^C\}$ parametrize a twisted Bethe eigenstate, [Belliard, Pakuliak, Ragoucy, Slavnov '12].
- We are interested in the case where just $\{\lambda^B\}, \{\mu^B\}$ satisfy Bethe equations, while $\{\lambda^C\}, \{\mu^C\}$ are free.

Graphical representation (1)



Graphical representation (2)



Sum expression for $SU(3)$ scalar product [Reshetikhin '86]

- To save space, let

$$f(\mu,\lambda)=\prod_{i=1}^{m}\prod_{j=1}^{\ell}f(\mu_i,\lambda_j)$$

for $\mu=\{\mu_1,\ldots,\mu_m\},\lambda=\{\lambda_1,\ldots,\lambda_\ell\}$.

- The scalar product can be expressed as $S(\{\mu^C\},\{\lambda^C\}|\{\lambda^B\},\{\mu^B\})=$

$$\sum\prod_{\lambda^B_{\mathrm{I}}}r_1(\lambda^B_{\mathrm{I}})\prod_{\lambda^C_{\mathrm{II}}}r_1(\lambda^C_{\mathrm{II}})\prod_{\mu^B_{\mathrm{II}}}r_2(\mu^B_{\mathrm{II}})\prod_{\mu^C_{\mathrm{I}}}r_2(\mu^C_{\mathrm{I}})f(\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}})f(\lambda^B_{\mathrm{II}},\lambda^B_{\mathrm{I}})f(\mu^C_{\mathrm{II}},\mu^C_{\mathrm{I}})f(\mu^B_{\mathrm{I}},\mu^B_{\mathrm{II}})$$

$$\times f(\mu^C_{\mathrm{I}},\lambda^C_{\mathrm{I}})f(\mu^B_{\mathrm{II}},\lambda^B_{\mathrm{II}})Z(\{\lambda^B_{\mathrm{II}}\},\{\mu^C_{\mathrm{I}}\}|\{\lambda^C_{\mathrm{II}}\},\{\mu^B_{\mathrm{I}}\})Z(\{\lambda^C_{\mathrm{I}}\},\{\mu^B_{\mathrm{II}}\}|\{\lambda^B_{\mathrm{I}}\},\{\mu^C_{\mathrm{II}}\})$$

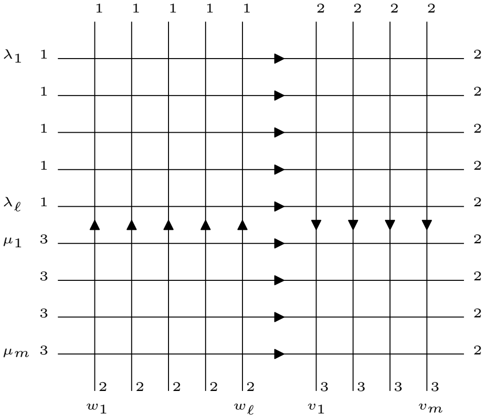
where $\sum$ is over

$\{\lambda^C\}=\{\lambda^C_{\mathrm{I}}\}\cup\{\lambda^C_{\mathrm{II}}\},\{\lambda^B\}=\{\lambda^B_{\mathrm{I}}\}\cup\{\lambda^B_{\mathrm{II}}\}$ such that $|\lambda^C_{\mathrm{I}}|=|\lambda^B_{\mathrm{I}}|,|\lambda^C_{\mathrm{II}}|=|\lambda^B_{\mathrm{II}}|$

$\{\mu^C\}=\{\mu^C_{\mathrm{I}}\}\cup\{\mu^C_{\mathrm{II}}\},\{\mu^B\}=\{\mu^B_{\mathrm{I}}\}\cup\{\mu^B_{\mathrm{II}}\}$ such that $|\mu^C_{\mathrm{I}}|=|\mu^B_{\mathrm{I}}|,|\mu^C_{\mathrm{II}}|=|\mu^B_{\mathrm{II}}|$

Reshetikhin's partition function [Reshetikhin '86]

- The quantity introduced in the sum above, $Z(\{\lambda\},\{\mu\}|\{w\},\{v\})$, is given by



- It is a candidate for the 'domain wall partition function' in the $SU(3)$ XXX spin-chain.

Reshetikhin's partition function

- It can be expressed as

$$Z(\{\lambda\},\{\mu\}|\{w\},\{v\}) = \sum \prod_{\mu_{\rm I},\mu_{\rm II}} f(\mu_{\rm I},\mu_{\rm II}) \prod_{\lambda_{\rm I},\lambda_{\rm II}} f(\lambda_{\rm II},\lambda_{\rm I}) \prod_{\mu_{\rm I},\lambda_{\rm I}} f(\mu_{\rm I},\lambda_{\rm I})$$
$$\times Z(\{\lambda_{\rm II}\}|\{\mu_{\rm II}\}) Z(\{\lambda_{\rm I}\}\cup\{\mu_{\rm II}\}|\{w\}) Z(\{v\}|\{\mu_{\rm I}\}\cup\{\lambda_{\rm II}\})$$

where $\sum$ is over

$$\{\lambda\} = \{\lambda_{\rm I}\}\cup\{\lambda_{\rm II}\}, \{\mu\} = \{\mu_{\rm I}\}\cup\{\mu_{\rm II}\} \text{ such that } |\lambda_{\rm II}| = |\mu_{\rm II}|$$

[W '12], [Belliard, Pakuliak, Ragoucy, Slavnov '12]

- Some limiting cases

$$Z(\{\lambda\}_\ell,\{\infty\}_m|\{w\}_\ell,\{v\}_m) = (-)^m Z(\{\lambda\}|\{w\})$$
$$Z(\{\infty\}_\ell,\{\mu\}_m|\{w\}_\ell,\{v\}_m) = Z(\{v\}|\{\mu\})$$
$$Z(\{\lambda\}_\ell,\{\mu\}_m|\{w\}_\ell,\{\infty\}_m) = \prod_{i=1}^{m}\prod_{j=1}^{\ell} f(\mu_i,w_j) Z(\{\lambda\}|\{w\})$$
$$Z(\{\lambda\}_\ell,\{\mu\}_m|\{\infty\}_\ell,\{v\}_m) = (-)^\ell \prod_{i=1}^{m}\prod_{j=1}^{\ell} f(v_i,\lambda_j) Z(\{v\}|\{\mu\})$$

Introducing the Bethe equations

- Returning to the scalar product, if $\{\lambda^B\}, \{\mu^B\}$ satisfy the Bethe equations, then $S(\{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\}) =$

$$\sum (-)^{|\lambda^B_{\mathrm{I}}| + |\mu^B_{\mathrm{II}}|} \prod_{\lambda^B_{\mathrm{I}}} \left(\prod_{j=1}^{\ell} \left(\frac{\lambda^B_{\mathrm{I}} - \lambda^B_j + 1}{\lambda^B_{\mathrm{I}} - \lambda^B_j - 1} \right) \prod_{k=1}^{m} f(\mu^B_k, \lambda^B_{\mathrm{I}}) \right) \times$$

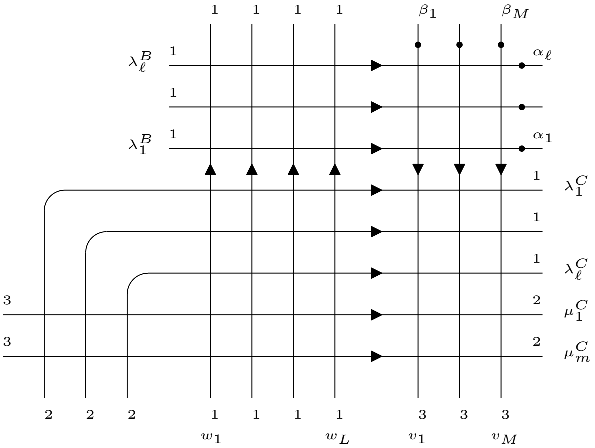
$$\prod_{\mu^B_{\mathrm{II}}} \left(\prod_{j=1}^{m} \left(\frac{\mu^B_{\mathrm{II}} - \mu^B_j + 1}{\mu^B_{\mathrm{II}} - \mu^B_j - 1} \right) \prod_{k=1}^{\ell} \frac{1}{f(\mu^B_{\mathrm{II}}, \lambda^B_k)} \right) \prod_{\lambda^C_{\mathrm{II}}} r_1(\lambda^C_{\mathrm{II}}) \prod_{\mu^C_{\mathrm{I}}} r_2(\mu^C_{\mathrm{I}}) \times$$

$$f(\lambda^C_{\mathrm{I}}, \lambda^C_{\mathrm{II}}) f(\lambda^B_{\mathrm{II}}, \lambda^B_{\mathrm{I}}) f(\mu^C_{\mathrm{II}}, \mu^C_{\mathrm{I}}) f(\mu^B_{\mathrm{I}}, \mu^B_{\mathrm{II}}) f(\mu^C_{\mathrm{I}}, \lambda^C_{\mathrm{I}}) f(\mu^B_{\mathrm{II}}, \lambda^B_{\mathrm{II}}) \times$$

$$Z(\{\lambda^B_{\mathrm{II}}\}, \{\mu^C_{\mathrm{I}}\} | \{\lambda^C_{\mathrm{II}}\}, \{\mu^B_{\mathrm{I}}\}) Z(\{\lambda^C_{\mathrm{I}}\}, \{\mu^B_{\mathrm{II}}\} | \{\lambda^B_{\mathrm{I}}\}, \{\mu^C_{\mathrm{II}}\})$$

- The goal is to sum this to a simpler expression, as was possible in the $SU(2)$ case.
- However, because the partition function Z is not known as a determinant, the way to do this is not obvious.

Taking the limit $\mu_1^B, \ldots, \mu_m^B \to \infty$



Taking the limit $\mu_1^B, \ldots, \mu_m^B \to \infty$

- Evaluating this limit directly from the sum form, we obtain

$$
\begin{aligned}
S(\{\mu^C\},\{\lambda^C\}|\{\lambda^B\},\{\infty\}) = \frac{1}{m!} \sum_{\substack{\{\lambda^C\}=\{\lambda^C_{\mathrm{I}}\}\cup\{\lambda^C_{\mathrm{II}}\} \\ \{\lambda^B\}=\{\lambda^B_{\mathrm{I}}\}\cup\{\lambda^B_{\mathrm{II}}\}}} \Bigg(\sum_{k=0}^{m} \sum_{|\mu^C_{\mathrm{I}}|=m-k,\, |\mu^C_{\mathrm{II}}|=k} (-)^{|\lambda^B_{\mathrm{I}}|} \binom{m}{k} \\
\times \prod_{\lambda^C_{\mathrm{II}}} r_1(\lambda^C_{\mathrm{II}}) \prod_{\mu^C_{\mathrm{I}}} r_2(\mu^C_{\mathrm{I}}) \prod_{\lambda^B_{\mathrm{I}}} \prod_{j=1}^{\ell} \left(\frac{\lambda^B_{\mathrm{I}} - \lambda^B_j + 1}{\lambda^B_{\mathrm{I}} - \lambda^B_j - 1} \right) f(\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}) f(\lambda^B_{\mathrm{II}},\lambda^B_{\mathrm{I}}) f(\mu^C_{\mathrm{II}},\mu^C_{\mathrm{I}}) f(\mu^C_{\mathrm{I}},\lambda^C_{\mathrm{I}}) \\
\times (m-k)!\, f(\mu^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}) Z(\{\lambda^B_{\mathrm{II}}\}|\{\lambda^C_{\mathrm{II}}\}) (-)^k k!\, Z(\{\lambda^C_{\mathrm{I}}\}|\{\lambda^B_{\mathrm{I}}\}) \Bigg)
\end{aligned}
$$

- In fact the above sum factorizes as

$$
\begin{aligned}
\Bigg(\sum (-)^{|\mu^C_{\mathrm{II}}|} \prod_{\mu^C_{\mathrm{I}}} \left(r_2(\mu^C_{\mathrm{I}}) \prod_{k=1}^{\ell} f(\mu^C_{\mathrm{I}},\lambda^C_k) \right) f(\mu^C_{\mathrm{II}},\mu^C_{\mathrm{I}}) \Bigg) \Bigg(\sum (-)^{|\lambda^B_{\mathrm{I}}|} \\
\prod_{\lambda^B_{\mathrm{I}}} \prod_{j=1}^{\ell} \left(\frac{\lambda^B_{\mathrm{I}} - \lambda^B_j + 1}{\lambda^B_{\mathrm{I}} - \lambda^B_j - 1} \right) \prod_{\lambda^C_{\mathrm{II}}} r_1(\lambda^C_{\mathrm{II}}) f(\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}) f(\lambda^B_{\mathrm{II}},\lambda^B_{\mathrm{I}}) Z(\{\lambda^B_{\mathrm{II}}\}|\{\lambda^C_{\mathrm{II}}\}) Z(\{\lambda^C_{\mathrm{I}}\}|\{\lambda^B_{\mathrm{I}}\}) \Bigg)
\end{aligned}
$$

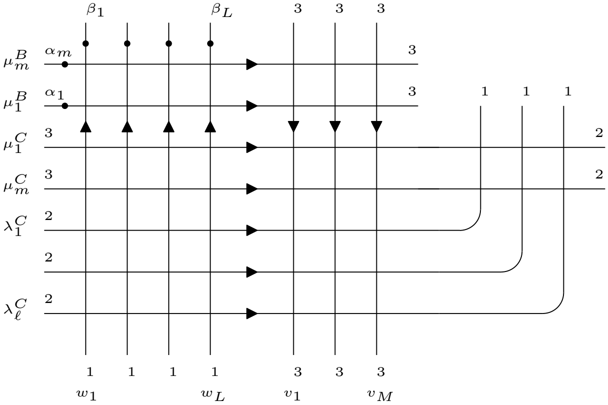
Result of the limit $\mu_1^B, \ldots, \mu_m^B \to \infty$

- Both of the factors are sums which we know how to evaluate. They are a partial domain wall partition function, and a Slavnov scalar product, from $SU(2)$ theory. Hence

$$S(\{\mu^C\},\{\lambda^C\}|\{\lambda^B\},\{\infty\}) =$$
$$\frac{\det\left((\mu_i^C)^{j-1} r_2(\mu_i^C) \prod_{k=1}^{\ell} \left(\frac{\mu_i^C - \lambda_k^C + 1}{\mu_i^C - \lambda_k^C} \right) - (\mu_i^C + 1)^{j-1} \right)_{1 \le i,j \le m}}{\prod_{1 \le i < j \le m} (\mu_j^C - \mu_i^C)} \times$$
$$\frac{\det\left(\frac{1}{\lambda_j^B - \lambda_i^C} \left(\prod_{k \ne j}^{\ell} (\lambda_k^B - \lambda_i^C + 1) r_1(\lambda_i^C) - \prod_{k \ne j}^{\ell} (\lambda_k^B - \lambda_i^C - 1) \right) \right)_{1 \le i,j \le \ell}}{\prod_{1 \le i < j \le \ell} (\lambda_j^C - \lambda_i^C)(\lambda_i^B - \lambda_j^B)}$$

[W '12]

Taking the limit $\lambda_1^B, \dots, \lambda_\ell^B \to \infty$



Taking the limit $\lambda^B_1,\ldots,\lambda^B_\ell \to \infty$

- Evaluating this limit directly from the sum form, we obtain

$$
\begin{aligned}
S(\{\mu^C\},\{\lambda^C\}|\{\infty\},\{\mu^B\}) = \frac{1}{\ell!} \sum_{\substack{\{\mu^C\}=\{\mu^C_{\mathrm{I}}\}\cup\{\mu^C_{\mathrm{II}}\} \\ \{\mu^B\}=\{\mu^B_{\mathrm{I}}\}\cup\{\mu^B_{\mathrm{II}}\}}} \Bigg(\sum_{k=0}^{\ell} \sum_{|\lambda^C_{\mathrm{I}}|=\ell-k,|\lambda^C_{\mathrm{II}}|=k} (-)^{|\mu^B_{\mathrm{II}}|} \binom{\ell}{k} \\
\times \prod_{\lambda^C_{\mathrm{II}}} r_1(\lambda^C_{\mathrm{II}}) \prod_{\mu^C_{\mathrm{I}}} r_2(\mu^C_{\mathrm{I}}) \prod_{\mu^B_{\mathrm{II}}} \prod_{j=1}^{m} \left(\frac{\mu^B_{\mathrm{II}}-\mu^B_j+1}{\mu^B_{\mathrm{II}}-\mu^B_j-1} \right) f(\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}) f(\mu^C_{\mathrm{II}},\mu^C_{\mathrm{I}}) f(\mu^B_{\mathrm{I}},\mu^B_{\mathrm{II}}) f(\mu^C_{\mathrm{I}},\lambda^C_{\mathrm{I}}) \\
\times k! Z(\{\mu^B_{\mathrm{I}}\}|\{\mu^C_{\mathrm{I}}\}) (-)^{\ell-k} (\ell-k)! f(\mu^C_{\mathrm{II}},\lambda^C_{\mathrm{I}}) Z(\{\mu^C_{\mathrm{II}}\}|\{\mu^B_{\mathrm{II}}\}) \Bigg)
\end{aligned}
$$

- Once again, the above sum has a clear factorization

$$
\begin{aligned}
f(\mu^C,\lambda^C) \left(\sum (-)^{|\lambda^C_{\mathrm{I}}|} \prod_{\lambda^C_{\mathrm{II}}} \left(r_1(\lambda^C_{\mathrm{II}}) \prod_{k=1}^{m} \frac{1}{f(\mu^C_k,\lambda^C_{\mathrm{II}})} \right) f(\lambda^C_{\mathrm{I}},\lambda^C_{\mathrm{II}}) \right) \Bigg(\sum (-)^{|\mu^B_{\mathrm{II}}|} \\
\prod_{\mu^B_{\mathrm{II}}} \prod_{j=1}^{m} \left(\frac{\mu^B_{\mathrm{II}}-\mu^B_j+1}{\mu^B_{\mathrm{II}}-\mu^B_j-1} \right) \prod_{\mu^C_{\mathrm{I}}} r_2(\mu^C_{\mathrm{I}}) f(\mu^C_{\mathrm{II}},\mu^C_{\mathrm{I}}) f(\mu^B_{\mathrm{I}},\mu^B_{\mathrm{II}}) Z(\{\mu^B_{\mathrm{I}}\}|\{\mu^C_{\mathrm{I}}\}) Z(\{\mu^C_{\mathrm{II}}\}|\{\mu^B_{\mathrm{II}}\}) \Bigg)
\end{aligned}
$$

Result of the limit $\lambda_1^B, \ldots, \lambda_\ell^B \to \infty$

- As before, both sums are known from $SU(2)$ results. Hence

$$
S(\{\mu^C\},\{\lambda^C\}|\{\infty\},\{\mu^B\}) =
$$

$$
\frac{\det\left((\lambda_i^C)^{j-1} r_1(\lambda_i^C) - (\lambda_i^C+1)^{j-1} \prod_{k=1}^{m} \left(\frac{\mu_k^C - \lambda_i^C + 1}{\mu_k^C - \lambda_i^C} \right) \right)_{1\leq i,j\leq \ell}}{\prod_{1\leq i<j\leq \ell} (\lambda_j^C - \lambda_i^C)} \times
$$

$$
\frac{\det\left(\frac{1}{\mu_j^B - \mu_i^C} \left(\prod_{k\neq j}^{m} (\mu_k^B - \mu_i^C + 1) r_2(\mu_i^C) - \prod_{k\neq j}^{m} (\mu_k^B - \mu_i^C - 1) \right) \right)_{1\leq i,j\leq m}}{\prod_{1\leq i<j\leq m} (\mu_j^C - \mu_i^C)(\mu_i^B - \mu_j^B)}
$$

[W '12]

Future work

- Can we use the two limits calculated here, as well as the work of [Reshetikhin '86], [Belliard, Pakuliak, Ragoucy, Slavnov '12], to find a manageable expression for the Bethe scalar product?

- Recently, it was shown in [Kostov, Matsuo '12], [Foda, W '12] that the XXX $SU(2)$ Bethe scalar product is in fact a partial domain wall partition function.

- Can a similar statement be made with respect to the XXX $SU(3)$ Bethe scalar product, and a 'partial' version of Reshetikhin's partition function?

- Generalization to $SU(n)$ models. The norm has already been conjectured in [Escobedo, Gromov, Sever, Vieira '11].