Algebraic Bethe ansatz approach to form factors and correlation functions of the cyclic solid-on-solid model

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Correlation functions in the ABA framework: first results

determinant representation for scalar products of Bethe states (Slavnov)
+ solution of the quantum inverse problem

\[ \leadsto \] determinant representation for form factors in finite volume
\[ \leadsto \] elementary building blocks of correlation functions as multiple sums
in finite volume and as multiple integrals in the thermodynamic limit

Two-point function:

sum up elementary blocks or form factors

\[ \leadsto \] Master equation representation for the finite chain:
N-fold multiple integral representation for the correlation function in finite volume

Asymptotic analysis of the two-point function

\[ \leadsto \] from the Master equation
\[ \leadsto \] from the series over form factors

Method essentially developed for XXZ chain or Quantum Bose gas
What about more complicated models?
A natural generalization of the XXZ Heisenberg chain is the XYZ chain:

$$H_{XYZ} = \sum_{m=1}^{N} \left\{ J_x \sigma^x_m \sigma^x_{m+1} + J_y \sigma^y_m \sigma^y_{m+1} + J_z \sigma^z_m \sigma^z_{m+1} \right\}$$

related to the 8-vertex model:

2-d square lattice model
link $\rightarrow \epsilon_j = \pm$
vertex $\rightarrow$ Boltzmann weight

$$R^{8V}(z_1/z_2)^{\epsilon_1,\epsilon_2}_{\epsilon'_1,\epsilon'_2} = \begin{pmatrix} a(z;p) & 0 & 0 & d(z;p) \\ 0 & b(z;p) & c(z;p) & 0 \\ 0 & c(z;p) & b(z;p) & 0 \\ d(z;p) & 0 & 0 & a(z;p) \end{pmatrix}$$

$z =$ spectral parameter
$p =$ elliptic parameter
$a, b, c, d =$ elliptic theta functions of $z$

No charge conservation through a vertex $\rightarrow$ no direct Bethe Ansatz solution
Baxter’s solution \( (Ann.Phys.73) \) $\rightarrow$ map onto an IRF model \( (8VSOS model) \)
eigenstates of 8V model given in terms of Bethe eigenstates of 8VSOS model
2-d square lattice model
vertex → local height $s_j$
$s_j - s_k = \pm 1$ (adjacent)
face → Boltzmann weight

$$R(u_i - \xi_j; s)^{\epsilon_i,\epsilon_j}_{\epsilon_i',\epsilon_j'} =$$

$$\begin{bmatrix}
s + \epsilon_j \\
\downarrow \quad u_i \\
s + \epsilon_j + \epsilon_i + \epsilon_j'
\end{bmatrix}$$

$$b(u; s) = \frac{[s+1][u]}{[s][u+1]}$$
$$c(u; s) = \frac{[s+u][1]}{[s][u+1]}$$
$$u = \text{spectral parameter}$$
$$s = \text{dynamical parameter}$$
$$[u] = \theta_1(\eta u; \tau) \quad p = e^{2\pi i \tau}$$

satisfying the **Dynamical Quantum Yang-Baxter Equation**:

$$R_{12}(u_1 - u_2; s + h_3) R_{13}(u_1; s) R_{23}(u_2; s + h_1)$$

$$= R_{23}(u_2; s) R_{13}(u_1; s + h_2) R_{12}(u_1 - u_2; s) \quad \text{with} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Charge conservation, solvable by Bethe Ansatz
ABA for the 8VSOS model

Felder, Varchenko (1996) : representations of $E_{\tau,\eta}(sl_2)$

- **Monodromy matrix:**

  \[ T_{a,1\ldots N}(u; \xi_1, \ldots, \xi_N; s) = R_{aN}(u - \xi_N; s + h_1 + \cdots + h_{N-1}) \cdots R_{a1}(u - \xi_1; s) \]

  \[ = \begin{pmatrix} A(u; s) & B(u; s) \\ C(u; s) & D(u; s) \end{pmatrix} \in \text{End}(\mathbb{C}^2 \otimes \mathcal{H}) \]

  \[ \hat{T}(u) = \begin{pmatrix} \hat{A}(u) & \hat{B}(u) \\ \hat{C}(u) & \hat{D}(u) \end{pmatrix} = T(u; \hat{s}) \begin{pmatrix} \hat{\tau}_s & 0 \\ 0 & \hat{\tau}_s^{-1} \end{pmatrix} \in \text{End}(\mathbb{C}^2 \otimes \text{Fun}(\mathcal{H})) , \]

  where $\hat{\tau}_s \hat{s} = (\hat{s} + 1) \hat{\tau}_s$, and the action of $\hat{s}$ and $\hat{\tau}_s$ on functions $f \in \text{Fun}(\mathcal{H})$ are given as $[\hat{s}f](s) = sf(s), \ [\hat{\tau}_s f](s) = f(s + 1)$.

- **Transfer matrix:**

  \[ \hat{t}(u) = \hat{A}(u) + \hat{D}(u) \]

  \[ \leadsto \text{preserve the space Fun}(\mathcal{H}[0]) \text{ of functions of the dynamical parameter } s \text{ with values in the zero-weight space } \mathcal{H}[0] \text{ of } \mathcal{H} \]

  \[ \leadsto [\hat{t}(u), \hat{t}(v)] = 0 \text{ on Fun}(\mathcal{H}[0]) \]

- **Space of states:** functions $\psi : s \mapsto \psi(s) \in \mathcal{H}[0]$

  - unrestricted case ($\eta$ generic): $s \in C_{s_0} = \{s_0 + j, j \in \mathbb{Z}\}$
  - cyclic case ($\eta = r/L$ rational): $s \in C_{s_0}^L = \{s_0 + j, j \in \mathbb{Z}/L\mathbb{Z}\}$
**reference state:**

\[ A(u; s) | 0 \rangle = \tilde{a}(u) | 0 \rangle, \quad D(u; s) | 0 \rangle = \frac{[s-1]}{[s+N-1]} \tilde{d}(u) | 0 \rangle \]

**Bethe states:** Suppose that the set of spectral parameters \( \{ v_1, \ldots, v_n \} \), satisfies the system of Bethe equations

\[ \tilde{a}(v_j) \prod_{l \neq j} \frac{[v_l - v_j + 1]}{[v_l - v_j]} = (-1)^r \omega^{-2} \tilde{d}(v_j) \prod_{l \neq j} \frac{[v_j - v_l + 1]}{[v_j - v_l]}, \quad j = 1, \ldots n, \]

with \( N = 2n + kL \) \((k \in \mathbb{Z})\) and \( \omega_L = (-1)^m \) (for \( \eta = r/L \)), then the state

\[ | \{ v \} \rangle : s \mapsto \varphi(s) B(v_1; s) B(v_2; s-1) \ldots B(v_n; s-n+1) | 0 \rangle \in \text{Fun}(\mathcal{H}[0]) \]

with \( \varphi(s) = \omega^s \prod_{j=1}^{n} \frac{[1]}{[s-j]} \)

is an eigenstate of the transfer matrix

\[ [\hat{t}(u) | \{ v \} \rangle] (s) = A(u; s) | \{ v \} \rangle (s+1) + D(u; s) | \{ v \} \rangle (s-1) \]

\[ = \tau(u; \{ v \}) | \{ v \} \rangle (s), \]

with eigenvalue

\[ \tau(u; \{ v \}) = \omega \tilde{a}(u) \prod_{l=1}^{n} \frac{[v_l - u + 1]}{[v_l - u]} + (-1)^r \omega^{-1} \tilde{d}(u) \prod_{l=1}^{n} \frac{[u - v_l + 1]}{[u - v_l]}. \]
Scalar product of Bethe states

Compute $\langle \{u\} | \{v\} \rangle$ in a compact and manageable form?

- **for XXZ:**
  
  - ∃ determinant representation for the scalar product when one of the state is a Bethe eigenstate (Slavnov, 1989)
  
  - this representation is related to Izergin’s determinant representation for the partition function with domain wall boundary conditions:
    
    $$Z_N(\{u\}; \{\xi\}) \propto \det_N \frac{\sinh \eta}{\sinh(u_i - \xi_j) \sinh(u_i - \xi_j + \eta)}$$

- **for SOS:**
  
  - no single determinant representation for the partition function with DWBC (Rosengren; Pakuliak, Rubtsov, Silantyev)
    
    $$Z_N(\{u\}; \{\xi\}; s) \propto \sum_{S \subset \{1, \ldots, N\}} (-1)^{|S|} \frac{[\gamma + s - |S|]}{[s - |S|]} \det_N \frac{[u_j - \xi^S_k + \gamma]}{[\gamma][u_j - \xi^S_k]}$$

  with $\xi^S_k = \begin{cases} \xi_k - 1 & \text{if } k \in S \\ \xi_k & \text{if } k \notin S \end{cases}$ (γ arbitrary).
Scalar product of Bethe states

Let \( \{u\}, \omega_u \) be solution of the Bethe equations and \( \{v\}, \omega_v \) be arbitrary, and consider the quantities:

- “partial scalar product” (general SOS model):

\[
S_n(\{u\}; \{v\}; s) = \langle 0 | C(u_n; s-n) \ldots C(u_1; s-1) B(v_1; s) \ldots B(v_n; s-n+1) | 0 \rangle
\]

\( \Rightarrow \) can be computed from Rosengren’s formula for the partition function with DWBC using the expressions of \( B \) and \( C \) in the \( F \)-basis (Maillet, Sanchez de Santos 96; Kitanine, Maillet, V.T. 99; Albert et al. 00)

- “total scalar product” (cyclic SOS):

\[
\langle \{u\} \mid \{v\} \rangle = \frac{1}{L} \sum_{s \in \mathbb{Z}/L \mathbb{Z}} \bar{\varphi}(s) \varphi(s) S_n(\{u\}; \{v\}; s)
\]

\( \Rightarrow \) The “total scalar product” (and the norm) can be expressed as a single determinant
Scalar product of Bethe states

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- sum of determinants (with more convenient representation in the cyclic case)

- "total scalar product" (cyclic SOS):

\[
\langle \{u\} | \{v\} \rangle = \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \bar{\varphi}(s) \varphi(s) S_n(\{u\}; \{v\}; s)
\]

\( \Rightarrow \) The "total scalar product" (and the norm) can be expressed as a single determinant
for generic $\eta$

$$S_n(\{u\}; \{v\}; s) \propto \sum_{S, \tilde{S} \subset \{1, \ldots, n\}} (-1)^{|s| + |\tilde{s}|} \frac{[\gamma + s - |S| + |\tilde{S}|]}{[s - |S| + |\tilde{S}|]}$$

$$\times \prod_{j \notin \tilde{S}} \left( \frac{\tilde{a}(v_j)}{\tilde{d}(v_j)} \right) \prod_{t=1}^{n} [u_t - v_j + 1] \prod_{j \in \tilde{S}} \left( \omega_u^{-2} \prod_{t=1}^{n} [u_t - v_j - 1] \right) \det_n \frac{[u_i - \xi_j^{S\tilde{S}} + \gamma]}{[\gamma][u_i - \xi_j^{S\tilde{S}}]}$$

with $\xi_k^{S\tilde{S}} = \begin{cases} 
\xi_k - 1 & \text{if } k \in S \text{ and } k \notin \tilde{S} \\
\xi_k + 1 & \text{if } k \notin S \text{ and } k \in \tilde{S} \\
\xi_k & \text{otherwise}
\end{cases}$ (\gamma \text{ arbitrary}).

for rational $\eta$ ($\eta = r/L$)

$$S_n(\{u\}; \{v\}; s) \propto \sum_{\ell=0}^{L-1} q^\ell s \left[ Ls_0 + \gamma + \ell \frac{\tau}{\eta} \right]_L \left[ 0' \right]_L \det_n \left[ \Omega^{(\ell)}_{\gamma}(\{u\}; \{v\}) \right],$$

with $q = e^{2\pi i \eta}$, $[u]_L = \theta_1(\eta u; L\tau)$, and

$$[\Omega^{(\ell)}_{\gamma}(\{u\}; \{v\})]_{ij} = \frac{1}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - q^{-\ell} \frac{[u_i - v_j + \gamma + 1]}{[u_i - v_j + 1]} \right\} \tilde{a}(v_j) \prod_{t=1}^{n} [u_t - v_j + 1]$$

$$+ \frac{(-1)^{rk}}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - q^{\ell} \frac{[u_i - v_j + \gamma - 1]}{[u_i - v_j - 1]} \right\} \omega_u^{-2} \tilde{d}(v_j) \prod_{t=1}^{n} [u_t - v_j - 1].$$
“Total scalar product” for rational $\eta$ ($\eta = r/L$)

Let $\{u\}, \omega_u$ be solution of the Bethe equations and $\{v\}, \omega_v$ be arbitrary

$$
\langle \{u\} | \{v\} \rangle = \left\{ \frac{1}{L} \sum_{s \in \mathbb{Z} + \mathbb{Z}/L\mathbb{Z}} \omega^s_v [\gamma + s] \omega^s_u [s] \right\} \prod_{t=1}^{n} \tilde{d}(u_t) \cdot \det_n [\Omega_\gamma(\{u\}; \{v\})] \prod_{j<k} [u_j - u_k][v_k - v_j],
$$

with $\gamma = -|u| + |v|$ and

$$
[\Omega_\gamma(\{u\}; \{v\})]_{ij} = \frac{1}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - \frac{\omega_v [u_i - v_j + \gamma + 1]}{\omega_u [u_i - v_j + 1]} \right\} \tilde{a}(v_j) \prod_{t=1}^{n} [u_t - v_j + 1] \\
+ \frac{(-1)^{rk}}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - \frac{\omega_u [u_i - v_j + \gamma - 1]}{\omega_v [u_i - v_j - 1]} \right\} \omega^{-2} u \tilde{d}(v_j) \prod_{t=1}^{n} [u_t - v_j - 1].
$$

Norm for rational $\eta$

$$
\langle \{u\} | \{u\} \rangle = \frac{1}{(-[0]^r)^n} \prod_{t=1}^{n} \tilde{d}(u_t) \cdot \det_n \left[ \frac{\partial}{\partial u_k} \gamma_{\omega_u}(u_j | \{u\}) \right]
$$

where

$$
\gamma_{\omega}(v | \{u\}) = \tilde{a}(v) \prod_{t=1}^{n} [u_t - v + 1] + (-1)^{rk} \omega^{-2} \tilde{d}(v) \prod_{t=1}^{n} [u_t - v - 1]
$$
Determinant representation for finite-size form factors

solution of the quantum inverse problem:

\[ E^{++}_i = \prod_{k=1}^{i-1} \hat{t}(\xi_k) \cdot \hat{A}(\xi_i) \cdot \prod_{k=i}^{1} \hat{t}(\xi_k)^{-1} \]

\[ E^{--}_i = \prod_{k=1}^{i-1} \hat{t}(\xi_k) \cdot \hat{D}(\xi_i) \cdot \prod_{k=i}^{1} \hat{t}(\xi_k)^{-1} \]

\[ \Rightarrow \text{express form factors in terms of scalar products} \]

\[ \Rightarrow \text{representation in terms of determinants:} \]

\[ \langle \{u\} | \sigma_i^Z | \{v\} \rangle = \left\{ \prod_{k=1}^{i-1} \frac{\tau(\xi_k, \{u\})}{\tau(\xi_k, \{v\})} \right\} \left\{ \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \omega_u^{-s} \omega_v^s \frac{[\gamma + s]}{[s]} \right\} \]

\[ \times \prod_{t=1}^{n} \tilde{d}(u_t) \prod_{k<l} [u_k - u_l][v_l - v_k] \det_n \left[ \Omega_\gamma(\{u\}; \{v\}) - 2P_\gamma(\{u\}; \{v\}|\xi_i) \right] \]

with \( \gamma = |v| - |u| \) and \( P_\gamma(\{u\}; \{v\}|\xi_i) \) is a rank 1 matrix

\[ \Rightarrow \text{possible to take the thermodynamic limit (computation of spontaneous staggered polarization)} \]
Conclusion and perspectives

Summary

- Determinant representations for scalar products/norms of Bethe states
- Determinant representations for finite-size form factors

Further questions . . .

- (multi-point) local height probabilities for CSOS model (work in progress)
  ~~~ use “partial scalar product”
  ~~~ sum of multiple integrals ?

- XYZ model ?
  ~~~ combinatorial complexity of Vertex-IRF transformation