Algebraic Bethe ansatz approach to form factors and correlation functions of the cyclic solid-on-solid model

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Algebraic Bethe ansatz approach to correlation functions

1. **Correlation functions in the ABA framework: first results**

   determinant representation for scalar products of Bethe states (Slavnov)

   $+$ solution of the quantum inverse problem

   - $\rightsquigarrow$ determinant representation for form factors in finite volume
   - $\rightsquigarrow$ elementary building blocks of correlation functions as multiple sums in finite volume and as multiple integrals in the thermodynamic limit

2. **Two-point function:** sum up elementary blocks or form factors

   - $\rightsquigarrow$ Master equation representation for the finite chain: N-fold multiple integral representation for the correlation function in finite volume

3. **Asymptotic analysis of the two-point function**

   - $\rightsquigarrow$ from the Master equation
   - $\rightsquigarrow$ from the series over form factors

Method essentially developed for XXZ chain or Quantum Bose gas

What about more complicated models ?

XYZ Heisenberg chain and 8VSOS model

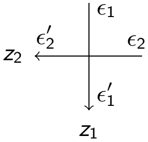
A natural generalization of the XXZ Heisenberg chain is the XYZ chain:

$$H_{XYZ} = \sum_{m=1}^{N} \left\{ J_x \sigma_m^x \sigma_{m+1}^x + J_y \sigma_m^y \sigma_{m+1}^y + J_z \sigma_m^z \sigma_{m+1}^z \right\}$$

related to the 8-vertex model:

2-d square lattice model
link $\longrightarrow \epsilon_j = \pm$
vertex $\longrightarrow$ Boltzmann weight

$$\mathbf{R}^{8V}(z_1/z_2)_{\epsilon_1',\epsilon_2'}^{\epsilon_1,\epsilon_2} =$$



$$\mathbf{R}^{8V}(z) = \begin{pmatrix} a(z;p) & 0 & 0 & d(z;p) \\ 0 & b(z;p) & c(z;p) & 0 \\ 0 & c(z;p) & b(z;p) & 0 \\ d(z;p) & 0 & 0 & a(z;p) \end{pmatrix}$$

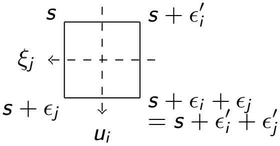
$z =$ spectral parameter
$p =$ elliptic parameter
$a, b, c, d =$ elliptic theta functions of z

No charge conservation through a vertex $\longrightarrow$ no direct Bethe Ansatz solution
Baxter's solution (Ann.Phys.73) $\longrightarrow$ map onto an IRF model (8VSOS model)
eigenstates of 8V model given in terms of Bethe eigenstates of 8VSOS model

8VSOS model

2-d square lattice model
vertex $\longrightarrow$ local height s_j
$\quad s_j - s_k = \pm 1$ (adjacent)
face $\longrightarrow$ Boltzmann weight

$$\mathbf{R}(u_i - \xi_j; s)^{\epsilon_i, \epsilon_j}_{\epsilon'_i, \epsilon'_j} =$$



$$\mathbf{R}(u; s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u;s) & c(u;s) & 0 \\ 0 & c(u;-s) & b(u;-s) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$b(u; s) = \frac{[s+1][u]}{[s][u+1]}$

$c(u; s) = \frac{[s+u][1]}{[s][u+1]}$

$u =$ spectral parameter

$s =$ dynamical parameter

$[u] = \theta_1(\eta u; \tau) \qquad p = e^{2\pi i \tau}$

satisfying the **Dynamical** Quantum Yang-Baxter Equation:

$$\mathbf{R}_{12}(u_1 - u_2; s + h_3)\, \mathbf{R}_{13}(u_1; s)\, \mathbf{R}_{23}(u_2; s + h_1)$$

$$= \mathbf{R}_{23}(u_2; s)\, \mathbf{R}_{13}(u_1; s + h_2)\, \mathbf{R}_{12}(u_1 - u_2; s) \qquad \text{with} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Charge conservation, solvable by Bethe Ansatz

ABA for the 8VSOS model

Felder, Varchenko (1996) : representations of $E_{\tau,\eta}(sl_2)$

- **Monodromy matrix:**

$$T_{a,1\dots N}(u;\xi_1,\dots,\xi_N;s) = R_{aN}(u-\xi_N;s+h_1+\dots+h_{N-1})\dots R_{a1}(u-\xi_1;s)$$
$$= \begin{pmatrix} A(u;s) & B(u;s) \\ C(u;s) & D(u;s) \end{pmatrix}_{[a]} \in \mathrm{End}(\mathbb{C}^2\otimes\mathcal{H})$$

$$\widehat{T}(u) = \begin{pmatrix} \widehat{A}(u) & \widehat{B}(u) \\ \widehat{C}(u) & \widehat{D}(u) \end{pmatrix}_{[a]} = T(u;\widehat{s}) \begin{pmatrix} \widehat{\tau}_s & 0 \\ 0 & \widehat{\tau}_s^{-1} \end{pmatrix}_{[a]} \in \mathrm{End}(\mathbb{C}^2\otimes\mathrm{Fun}(\mathcal{H})),$$

where $\widehat{\tau}_s\,\widehat{s} = (\widehat{s}+1)\,\widehat{\tau}_s$, and the action of $\widehat{s}$ and $\widehat{\tau}_s$ on functions $f\in\mathrm{Fun}(\mathcal{H})$ are given as $[\widehat{s}f](s) = sf(s)$, $[\widehat{\tau}_s f](s) = f(s+1)$.

- **Transfer matrix:** $\widehat{t}(u) = \widehat{A}(u) + \widehat{D}(u)$

  $\rightsquigarrow$ preserve the space $\mathrm{Fun}(\mathcal{H}[0])$ of functions of the dynamical parameter s with values in the zero-weight space $\mathcal{H}[0]$ of $\mathcal{H}$

  $\rightsquigarrow$ $[\widehat{t}(u),\widehat{t}(v)] = 0$ on $\mathrm{Fun}(\mathcal{H}[0])$

- **Space of states:** functions $\psi : s \mapsto \psi(s) \in \mathcal{H}[0]$
  - unrestricted case (η generic): $s \in \mathbf{C}_{s_0} = \{s_0+j, j\in\mathbb{Z}\}$
  - cyclic case ($\eta = r/L$ rational): $s \in \mathbf{C}^{\mathbf{L}}_{s_0} = \{s_0+j, j\in\mathbb{Z}/L\mathbb{Z}\}$

- **reference state:**
  $A(u;s)|\,0\,\rangle = \tilde{a}(u)|\,0\,\rangle, \qquad D(u;s)|\,0\,\rangle = \frac{[s-1]}{[s+N-1]}\tilde{d}(u)|\,0\,\rangle$
- **Bethe states:** Suppose that the set of spectral parameters $\{v_1,\dots,v_n\}$, satisfies the system of Bethe equations

$$\tilde{a}(v_j)\prod_{l\neq j}\frac{[v_l-v_j+1]}{[v_l-v_j]} = (-1)^{rk}\omega^{-2}\,\tilde{d}(v_j)\prod_{l\neq j}\frac{[v_j-v_l+1]}{[v_j-v_l]}, \quad j=1,\dots n,$$

with $N=2n+kL$ $(k\in\mathbb{Z})$ and $\omega^L=(-1)^{rn}$ (for $\eta=r/L$), then the state

$$|\,\{v\}\,\rangle : s\mapsto \varphi(s)B(v_1;s)B(v_2;s-1)\dots B(v_n;s-n+1)|\,0\,\rangle \in \mathrm{Fun}(\mathcal{H}[0])$$

with $\varphi(s)=\omega^s\prod_{j=1}^n\frac{[1]}{[s-j]}$

is an eigenstate of the transfer matrix

$$\begin{aligned}\big[\widehat{t}(u)\,|\,\{v\}\,\rangle\big](s) &= A(u;s)\,|\,\{v\}\,\rangle(s+1)+D(u;s)\,|\,\{v\}\,\rangle(s-1)\\ &= \tau(u;\{v\})\,|\,\{v\}\,\rangle(s),\end{aligned}$$

with eigenvalue

$$\tau(u;\{v\}) = \omega\,\tilde{a}(u)\prod_{l=1}^n\frac{[v_l-u+1]}{[v_l-u]} + (-1)^{rk}\omega^{-1}\,\tilde{d}(u)\prod_{l=1}^n\frac{[u-v_l+1]}{[u-v_l]}.$$

Scalar product of Bethe states

Compute $\langle \{u\} | \{v\} \rangle$ in a compact and manageable form ?

- **for XXZ:**
  - $\exists$ determinant representation for the scalar product when one of the state is a Bethe eigenstate (Slavnov, 1989)
  - this representation is related to Izergin's determinant representation for the partition function with domain wall boundary conditions:

$$Z_N(\{u\};\{\xi\}) \propto \det_N \frac{\sinh \eta}{\sinh(u_i - \xi_j)\sinh(u_i - \xi_j + \eta)}$$

- **for SOS:**
  - no single determinant representation for the partition function with DWBC (Rosengren; Pakuliak, Rubtsov, Silantyev)

$$Z_N(\{u\};\{\xi\};s) \propto \sum_{S\subset\{1,\ldots,N\}} (-1)^{|S|} \frac{[\gamma + s - |S|]}{[s-|S|]} \det_N \frac{[u_j - \xi_k^S + \gamma]}{[\gamma][u_j - \xi_k^S]}$$

with $\xi_k^S = \begin{cases} \xi_k - 1 & \text{if} \quad k \in S \\ \xi_k & \text{if} \quad k \notin S \end{cases}$ (γ arbitrary).

Scalar product of Bethe states

Let $\{u\}, \omega_u$ be solution of the Bethe equations and $\{v\}, \omega_v$ be arbitrary, and consider the quantities:

- "partial scalar product" (general SOS model):

$$S_n(\{u\};\{v\};s) = \langle\, 0\,|C(u_n;s-n)\dots C(u_1;s-1)B(v_1;s)\dots B(v_n;s-n+1)|\,0\,\rangle$$

  - ⇝ can be computed from Rosengren's formula for the partition function with DWBC using the expressions of B and C in the F-basis (Maillet, Sanchez de Santos 96; Kitanine,Maillet,V.T. 99; Albert et al. 00)
  - ⇝ sum of determinants (with more convenient representation in the cyclic case)

- "total scalar product" (cyclic SOS):

$$\langle\{u\}\,|\,\{v\}\rangle = \frac{1}{L}\sum_{s\in s_0+\mathbb{Z}/L\mathbb{Z}} \bar{\varphi}(s)\,\varphi(s)\,S_n(\{u\};\{v\};s)$$

  - ⇝ The "total scalar product" (and the norm) can be expressed as a single determinant

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- "total scalar product" (cyclic SOS):

$$\langle \{u\} \,|\, \{v\} \rangle = \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \bar{\varphi}(s)\, \varphi(s)\, S_n(\{u\};\{v\};s)$$

  - ⇝ The "total scalar product" (and the norm) can be expressed as a single determinant

for generic η

$$S_n(\{u\};\{v\};s) \propto \sum_{S,\tilde{S}\subset\{1,\dots,n\}} (-1)^{|S|+|\tilde{S}|} \frac{[\gamma+s-|S|+|\tilde{S}|]}{[s-|S|+|\tilde{S}|]}$$

$$\times \prod_{j\notin\tilde{S}} \left\{ \frac{\tilde{a}(v_j)}{\tilde{d}(v_j)} \prod_{t=1}^{n}[u_t-v_j+1] \right\} \prod_{j\in\tilde{S}} \left\{ \omega_u^{-2} \prod_{t=1}^{n}[u_t-v_j-1] \right\} \det_n \frac{[u_i-\xi_j^{S\tilde{S}}+\gamma]}{[\gamma][u_i-\xi_j^{S\tilde{S}}]}$$

$$\text{with } \xi_k^{S\tilde{S}} = \begin{cases} \xi_k - 1 & \text{if } k\in S \text{ and } k\notin\tilde{S} \\ \xi_k + 1 & \text{if } k\notin S \text{ and } k\in\tilde{S} \\ \xi_k & \text{otherwise} \end{cases} \qquad (\gamma \text{ arbitrary}).$$

for rational η ($\eta = r/L$)

$$S_n(\{u\};\{v\};s) \propto \sum_{\ell=0}^{L-1} q^{\ell s} \frac{[Ls_0+\gamma+\ell\frac{\tau}{\eta}]_L\,[0]'_L}{[Ls_0]_L\,[\gamma+\ell\frac{\tau}{\eta}]_L} \det_n\left[\Omega_\gamma^{(\ell)}(\{u\};\{v\})\right],$$

with $q = e^{2\pi i\eta}$, $[u]_L = \theta_1(\eta u; L\tau)$, and

$$\left[\Omega_\gamma^{(\ell)}(\{u\};\{v\})\right]_{ij} = \frac{1}{[\gamma]} \left\{ \frac{[u_i-v_j+\gamma]}{[u_i-v_j]} - q^{-\ell}\frac{[u_i-v_j+\gamma+1]}{[u_i-v_j+1]} \right\} \tilde{a}(v_j)\prod_{t=1}^{n}[u_t-v_j+1]$$

$$+\frac{(-1)^{rk}}{[\gamma]} \left\{ \frac{[u_i-v_j+\gamma]}{[u_i-v_j]} - q^{\ell}\frac{[u_i-v_j+\gamma-1]}{[u_i-v_j-1]} \right\} \omega_u^{-2}\tilde{d}(v_j)\prod_{t=1}^{n}[u_t-v_j-1].$$

"Total scalar product" for rational η ($\eta = r/L$)

Let $\{u\}, \omega_u$ be solution of the Bethe equations and $\{v\}, \omega_v$ be arbitrary

$$\langle \{u\} \,|\, \{v\} \rangle = \left\{ \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \frac{\omega_v^s}{\omega_u^s} \frac{[\gamma + s]}{[s]} \right\} \frac{\prod_{t=1}^n \tilde{d}(u_t) \cdot \det_n \left[\Omega_\gamma(\{u\};\{v\}) \right]}{\prod_{j<k} [u_j - u_k][v_k - v_j]},$$

with $\gamma = -|u| + |v|$ and

$$[\Omega_\gamma(\{u\};\{v\})]_{ij} = \frac{1}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - \frac{\omega_v}{\omega_u} \frac{[u_i - v_j + \gamma + 1]}{[u_i - v_j + 1]} \right\} \tilde{a}(v_j) \prod_{t=1}^n [u_t - v_j + 1]$$
$$+ \frac{(-1)^{rk}}{[\gamma]} \left\{ \frac{[u_i - v_j + \gamma]}{[u_i - v_j]} - \frac{\omega_u}{\omega_v} \frac{[u_i - v_j + \gamma - 1]}{[u_i - v_j - 1]} \right\} \omega_u^{-2} \tilde{d}(v_j) \prod_{t=1}^n [u_t - v_j - 1].$$

Norm for rational η

$$\langle \{u\} \,|\, \{u\} \rangle = \frac{1}{(-[0]')^n} \frac{\prod_{t=1}^n \tilde{d}(u_t)}{\prod_{j \neq k} [u_j - u_k]} \cdot \det_n \left[\frac{\partial}{\partial u_k} \mathcal{Y}_{\omega_u}(u_j | \{u\}) \right]$$

where

$$\mathcal{Y}_\omega(v|\{u\}) = \tilde{a}(v) \prod_{t=1}^n [u_t - v + 1] + (-1)^{rk} \omega^{-2} \tilde{d}(v) \prod_{t=1}^n [u_t - v - 1]$$

Determinant representation for finite-size form factors

solution of the quantum inverse problem:

$$E_i^{++} = \prod_{k=1}^{i-1} \widehat{t}(\xi_k) \cdot \widehat{A}(\xi_i) \cdot \prod_{k=i}^{1} \left[\widehat{t}(\xi_k)\right]^{-1}$$

$$E_i^{--} = \prod_{k=1}^{i-1} \widehat{t}(\xi_k) \cdot \widehat{D}(\xi_i) \cdot \prod_{k=i}^{1} \left[\widehat{t}(\xi_k)\right]^{-1}$$

- $\rightsquigarrow$ express form factors in terms of scalar products
- $\rightsquigarrow$ representation in terms of determinants:

$$\langle \{u\} \,|\, \sigma_i^z \,|\, \{v\} \rangle = \left\{ \prod_{k=1}^{i-1} \frac{\tau(\xi_k, \{u\})}{\tau(\xi_k, \{v\})} \right\} \left\{ \frac{1}{L} \sum_{s \in s_0 + \mathbb{Z}/L\mathbb{Z}} \omega_u^{-s} \omega_v^{s} \frac{[\gamma + s]}{[s]} \right\}$$

$$\times \frac{\prod_{t=1}^{n} \tilde{d}(u_t)}{\prod_{k<l} [u_k - u_l][v_l - v_k]} \det_n \left[\Omega_\gamma(\{u\}; \{v\}) - 2\mathcal{P}_\gamma(\{u\}; \{v\} | \xi_i) \right]$$

with $\gamma = |v| - |u|$ and $\mathcal{P}_\gamma(\{u\}; \{v\} | \xi_i)$ is a rank 1 matrix

- $\rightsquigarrow$ possible to take the thermodynamic limit (computation of spontaneous staggered polarization)

Conclusion and perspectives

- **Summary**
  - ⋆ Determinant representations for scalar products/norms of Bethe states
  - ⋆ Determinant representations for finite-size form factors

- **Further questions . . .**
  - ⋆ (multi-point) local height probabilities for CSOS model (work in progress)
    - ⇝ use "partial scalar product"
    - ⇝ sum of multiple integrals ?
  - ⋆ **XYZ model ?**
    - ⇝ combinatorial complexity of Vertex-IRF transformation