Algebraic Bethe Ansatz for scalar products in SU(3)-invariant models

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in Collaboration with

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Algebraic Bethe Ansatz for scalar products in SU(3)-invariant models

S. Belliard, S. Pakuliak, E. Ragoucy, N.S.

[1] *Highest coefficient of scalar products in SU(3)-invariant integrable models, arXiv:1206.4931*

[2] *Algebraic Bethe ansatz for scalar products in SU(3)-invariant integrable models, arXiv:1207.0956*

Problem

Suppose that we have a quantum model and we want to calculate matrix elements of some operator $\widehat{\mathcal{O}}$

$$\mathcal{O}_{\psi,\psi'} = \langle\psi|\widehat{\mathcal{O}}|\psi'\rangle$$

where $|\psi\rangle$ and $|\psi'\rangle$ are eigenstates of the Hamiltonian.

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where $|\psi\rangle$ and $|\psi'\rangle$ are eigenstates of the Hamiltonian.

Suppose that the action of $\widehat{\mathcal{O}}$ to the right or to the left is known

$$\widehat{\mathcal{O}}|\psi'\rangle = |\phi'\rangle \qquad \langle\psi|\widehat{\mathcal{O}} = \langle\phi|$$

Then we reduce the problem to the calculation of the scalar product, where one of the states is the eigenstate of the Hamiltonian.

$$\mathcal{O}_{\psi,\psi'} = \langle\psi|\phi'\rangle \qquad \mathcal{O}_{\psi,\psi'} = \langle\phi|\psi'\rangle$$

- Quantum Inverse Scattering Method

(L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan '79)

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$\mathfrak{gl}_2$-based models (rational and trigonometric R-matrix)

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- Scalar products $\langle\phi|\psi\rangle$

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In order to apply the scalar product formula $\langle\phi|\psi\rangle$ to the calculation of form factors and correlation functions one should know the action of local operators on Bethe vectors $|\psi\rangle$.

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- Quantum Inverse Scattering Problem

N. Kitanine, J.M. Maillet, V. Terras '99

J.M. Maillet, V. Terras '00 (Higher rank algebras)

$$\widehat{\mathcal{O}}|\psi\rangle = |\phi\rangle$$

Higher rank algebras

- Nested Bethe ansatz

(P.P. Kulish, N.Yu. Reshetikhin, '83)

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V.O. Tarasov, A.N. Varchenko '95

S. Belliard, S. Khoroshkin, S. Pakuliak, E. Ragoucy '08, '10

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S. Belliard, S. Pakuliak, E. Ragoucy '10

J. Escobedo, N. Gromov, A. Sever, P. Vieira '11

B. Pozsgay, W.-V. van G. Oei, M. Kormos '12

M. Wheeler '12

Algebraic Bethe Ansatz

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v)$$

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Algebraic Bethe Ansatz works if there exists a pseudovacuum vector $|0\rangle$ and dual pseudovacuum vector $\langle 0|$

$$T_{jj}(u)|0\rangle = r_j(u)|0\rangle, \qquad T_{jk}(u)|0\rangle = 0, \quad j > k$$

$$\langle 0|T_{jj}(u) = r_j(u)\langle 0|, \qquad \langle 0|T_{jk}(u) = 0, \quad j < k$$

One can set one of $r_j(u)$ equals to 1 without loss of generality. Other $r_j(u)$ remain free functional parameters (generalized model).

Bethe vectors

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We say that $|\psi(\bar{u})\rangle$ is an off-shell Bethe vector, if the parameters $\bar{u}$ are generic complex numbers.

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We say that $|\psi(\bar{u})\rangle$ is an off-shell Bethe vector, if the parameters $\bar{u}$ are generic complex numbers.

If the parameters $\bar{u}$ satisfy the system of Bethe equations, then we say that $|\psi(\bar{u})\rangle$ is an on-shell Bethe vector

$$\mathcal{T}(w)|\psi(\bar{u})\rangle = \tau(w|\bar{u})|\psi(\bar{u})\rangle$$

The rules of the game

What is given:

- The R-matrix (thus, the commutation relations $[T_{ij}(u), T_{k\ell}(v)]$).
- The action $T_{ij}(u)|0\rangle$ for $i \leq j$.
- The form of the Bethe vectors $|\psi(\bar{u})\rangle = P\Big(T_{ij}(u_k)\Big)|0\rangle, \qquad i < j$

  (that is the polynomials $P\Big(T_{ij}(u_k)\Big)$ are fixed).

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  (that is the polynomials $P\Big(T_{ij}(u_k)\Big)$ are fixed).

What we want to find:

$$\langle\psi(\bar{u}^C)|\psi(\bar{u}^B)\rangle = \langle 0|P\Big(T_{k\ell}(u_k^C)\Big)P\Big(T_{ij}(u_k^B)\Big)|0\rangle, \qquad k > \ell, \quad i < j$$

We consider R-matrix of the form

$$R(u-v) = \mathbf{I} + g(u,v)\mathbf{P}, \qquad g(u,v) = \frac{c}{u-v}$$

Other rational functions often appearing in the formulas

$$f(u,v) = 1 + g(u,v)$$

$$h(u,v) = \frac{f(u,v)}{g(u,v)}$$

$$t(u,v) = \frac{g(u,v)}{h(u,v)}$$

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$$f(u,v) = 1 + g(u,v) = \frac{u-v+c}{u-v}$$

$$h(u,v) = \frac{f(u,v)}{g(u,v)} = \frac{u-v+c}{c}$$

$$t(u,v) = \frac{g(u,v)}{h(u,v)} = \frac{c^2}{(u-v+c)(u-v)}$$

Shorthand notations for products

$$T_{ij}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{ij}(w_k)$$

$$r_1(\bar{u}^C_{\mathrm{II}}) = \prod_{u^C_k \in \bar{u}^C_{\mathrm{II}}} r_1(u^C_k)$$

$$h(\bar{u}^C, v_j) = \prod_{u^C_k \in \bar{u}^C} h(u^C_k, v_j)$$

$$f(\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}}) = \prod_{u^B_j \in \bar{u}^B_{\mathrm{I}}} \prod_{u^B_k \in \bar{u}^B_{\mathrm{II}}} f(u^B_j, u^B_k)$$

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$$\Delta_n(\bar{u}) = \prod_{j>k}^{n} g(u_j, u_k), \qquad \Delta'_n(\bar{u}) = \prod_{j<k}^{n} g(u_j, u_k)$$

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Special subsets

$$\bar{u}_j = \bar{u} \setminus u_j, \qquad \bar{v}^C_k = \bar{v}^C \setminus v^C_k, \ \ldots$$

Algebraic Bethe Ansatz for SU(2) models

$$T(w) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$A(w)|0\rangle = r(w)|0\rangle, \qquad D(w)|0\rangle = |0\rangle, \qquad C(w)|0\rangle = 0$$

$$\langle 0|A(w) = r(w)\langle 0|, \qquad \langle 0|D(w) = \langle 0|, \qquad \langle 0|B(w) = 0$$

$r(w)$ is a free functional parameter.

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Bethe vectors

$$|\psi_n(\bar{u})\rangle = B(\bar{u})|0\rangle, \qquad \langle\psi_n(\bar{u})| = \langle 0|C(\bar{u}), \qquad \bar{u} = u_1, \ldots, u_n$$

On-shell Bethe vectors

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If the set $\bar{u}$ satisfies Bethe equations

$$r(u_k) = \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)}, \qquad k = 1, \ldots, n \qquad \bar{u}_k = \bar{u} \setminus u_k$$

then

$$\mathcal{T}(w)|\psi_n(\bar{u})\rangle = \tau(w|\bar{u})|\psi_n(\bar{u})\rangle$$

$$\tau(w|\bar{u}) = r(w)f(\bar{u}, w) + f(w, \bar{u})$$

Algebraic Bethe Ansatz for SU(3) models

$$T_{11}(w)|0\rangle = r_1(w)|0\rangle, \qquad T_{22}(w)|0\rangle = |0\rangle, \qquad T_{33}(w)|0\rangle = r_3(w)|0\rangle$$

$$T_{jk}(w)|0\rangle = 0, \quad j > k; \qquad \langle 0|T_{jk}(w) = 0, \quad j < k$$

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Bethe vectors

$$|\psi_{a,b}(\bar{u};\bar{v})\rangle = P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big)|0\rangle, \quad i < j,$$

$$\langle\psi_{a,b}(\bar{u};\bar{v})| = \langle 0|P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big), \quad i > j,$$

$$\bar{u} = u_1, \ldots, u_a$$

$$\bar{v} = v_1, \ldots, v_b$$

$$|\psi_{1,1}(u;v)\rangle = T_{12}(u)T_{23}(v)|0\rangle + g(v,u)T_{13}(u)|0\rangle$$

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$$\mathcal{T}(w)|\psi_{a,b}(\bar{u};\bar{v})\rangle = \tau(w|\bar{u},\bar{v})|\psi_{a,b}(\bar{u};\bar{v})\rangle$$

$$\tau(w|\bar{u},\bar{v}) = r_1(w)f(\bar{u},w) + f(w,\bar{u})f(\bar{v},w) + r_3(w)f(w,\bar{v})$$

Bethe equations

$$r_1(u_k) = \frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)}f(\bar{v},u_k), \qquad r_3(v_k) = \frac{f(\bar{v}_k,v_k)}{f(v_k,\bar{v}_k)}f(v_k,\bar{u})$$

$$\bar{u}_k = \bar{u} \setminus u_k, \qquad \bar{v}_k = \bar{v} \setminus v_k$$

Twisted transfer matrix

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v)$$

Let $\widetilde{T}(w) = \widehat{\kappa}T(w)$, where $[\widehat{\kappa}_1\widehat{\kappa}_2, R_{12}] = 0$

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$$\widehat{\kappa} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

κ is called twist parameter

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$$R_{12}(u,v)\widetilde{T}_1(u)\widetilde{T}_2(v) = \widetilde{T}_2(v)\widetilde{T}_1(u)R_{12}(u,v)$$

Twisted on-shell vectors

$$\mathcal{T}_\kappa(w) = \mathrm{tr}\,\widetilde{T}(w) \qquad\qquad \mathcal{T}_\kappa(w)|\psi^{(\kappa)}_{a,b}(\bar{u};\bar{v})\rangle = \tau_\kappa(w)|\psi^{(\kappa)}_{a,b}(\bar{u};\bar{v})\rangle$$

$$\tau_\kappa(w) \equiv \tau_\kappa(w|\bar{u},\bar{v}) = r_1(w)f(\bar{u},w) + \kappa f(w,\bar{u})f(\bar{v},w) + r_3(w)f(w,\bar{v})$$

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$$r_1(u_k) = \kappa\,\frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)}f(\bar{v},u_k), \qquad r_3(v_k) = \kappa\,\frac{f(\bar{v}_k,v_k)}{f(v_k,\bar{v}_k)}f(v_k,\bar{u})$$

Scalar product in SU(2)-invariant models

Izergin–Korepin representation

$$\mathcal{S}_N \equiv \mathcal{S}_N(\bar{u}^C; \bar{u}^B) = \langle \psi_n(\bar{u}^C) | \psi_n(\bar{u}^B) \rangle$$

Scalar product in SU(2)-invariant models

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$$\mathcal{S}_N = \sum r(\bar{u}^B_{\mathrm{II}}) r(\bar{u}^C_{\mathrm{I}}) K_n(\bar{u}^B_{\mathrm{I}} | \bar{u}^C_{\mathrm{I}}) K_{N-n}(\bar{u}^C_{\mathrm{II}} | \bar{u}^B_{\mathrm{II}}) f(\bar{u}^C_{\mathrm{II}}, \bar{u}^C_{\mathrm{I}}) f(\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}})$$

The sum is taken over partitions of the sets $\bar{u}^C$ and $\bar{u}^B$

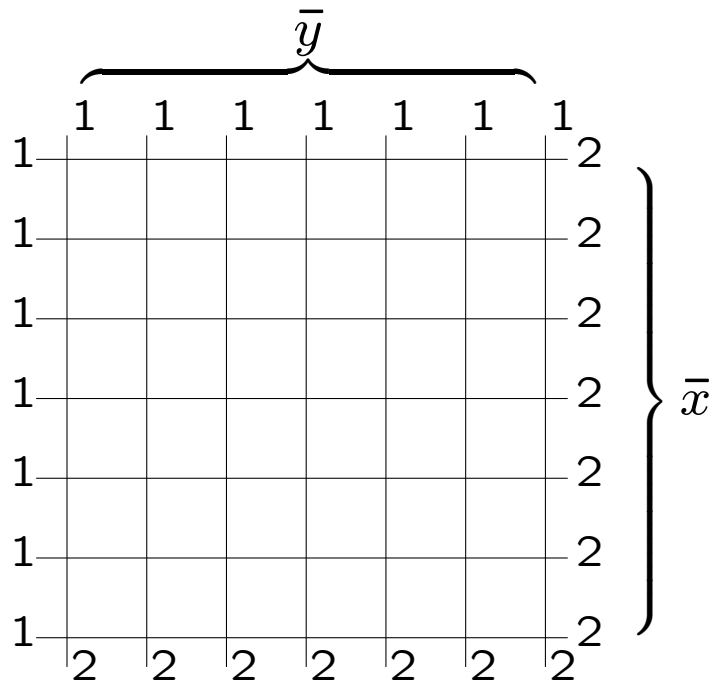
$$\bar{u}^B = \{\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}}\}$$

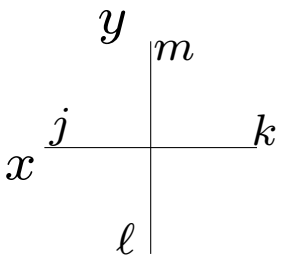
$$\#\bar{u}^C_{\mathrm{I}} = \#\bar{u}^B_{\mathrm{I}} = n, \qquad n = 0, 1, \ldots, N$$

$$\bar{u}^C = \{\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}\}$$

Domain Wall Partition Function (DWPF)

$K_n(\bar{x}|\bar{y}) \quad =$





$= \Big(R(x,y)\Big)_{jk,\ell m}$

Domain Wall Partition Function (DWPF)

Determinant representation (Izergin '87)

$$K_n(\bar{x}|\bar{y}) = \Delta_n(\bar{x})\Delta'_n(\bar{y})h(\bar{x},\bar{y}) \det_n \; t(x_j, y_k)$$

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$$K_n(\bar{x}|\bar{y}) = \Delta_n(\bar{x})\Delta'_n(\bar{y})h(\bar{x},\bar{y}) \det_n \; t(x_j, y_k)$$

$$K_n(\bar{x}|\bar{y}) = \frac{\prod\limits_{j,k=1}^{n}(x_j - y_k + c)}{\prod\limits_{j>k}^{n}(x_j - x_k)(y_k - y_j)} \det_n \left[\frac{c}{(x_j - y_k)(x_j - y_k + c)}\right]$$

Scalar product in SU(2)-invariant models

$$\mathcal{S}_N = \langle \psi_n(\bar{u}^C) | \psi_n(\bar{u}^B) \rangle$$

Suppose that $|\psi_n(\bar{u}^B)\rangle$ is on-shell vector, i.e.
the set $\bar{u}^B$ satisfies Bethe equations

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the set $\bar{u}^B$ satisfies Bethe equations

$$\mathcal{S}_N = \Delta_N(\bar{u}^C)\Delta'_N(\bar{u}^B) \det_N \mathcal{M}_{jk}$$

$$\mathcal{M}_{jk} = t(u_k^C, u_j^B) h(u_k^C, \bar{u}^B) - (-1)^N r(u_k^C) t(u_j^B, u_k^C) h(\bar{u}^B, u_k^C)$$

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It is easy to check that the entries $\mathcal{M}_{jk}$ are proportional to the Jacobian of the transfer matrix eigenvalue

$$\mathcal{M}_{jk} = cg^{-1}(u_k^C, \bar{u}^B) \cdot \frac{\partial \tau(u_k^C|\bar{u}^B)}{\partial u_j^B}$$

Determinant formula

$$\mathcal{S}_N = \Delta_N(\bar{u}^C)\Delta'_N(\bar{u}^B)\det_N\left[cg^{-1}(u_k^C,\bar{u}^B)\,\frac{\partial\tau(u_k^C|\bar{u}^B)}{\partial u_j^B}\right]$$

Izergin–Korepin formula

$$\mathcal{S}_N = \sum r(\bar{u}_{\mathrm{II}}^B)r(\bar{u}_{\mathrm{I}}^C)K_n(\bar{u}_{\mathrm{I}}^B|\bar{u}_{\mathrm{I}}^C)K_{N-n}(\bar{u}_{\mathrm{II}}^C|\bar{u}_{\mathrm{II}}^B)f(\bar{u}_{\mathrm{I}}^B,\bar{u}_{\mathrm{II}}^B)f(\bar{u}_{\mathrm{II}}^C,\bar{u}_{\mathrm{I}}^C)$$

Determinant formula

$$\mathcal{S}_N = \Delta_N(\bar{u}^C)\Delta'_N(\bar{u}^B)\,\det_N\Big[cg^{-1}(u_k^C,\bar{u}^B)\,\frac{\partial\tau(u_k^C|\bar{u}^B)}{\partial u_j^B}\Big]$$

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$$r(u_k^C) = 0$$

$$r(u_k^C) \to \infty$$

Determinant formula

$$\mathcal{S}_N = \Delta_N(\bar{u}^C)\Delta'_N(\bar{u}^B)\det_N\Big[t(u_k^C, u_j^B)h(u_k^C, \bar{u}^B)\Big]$$

Izergin–Korepin formula

$$\bar{u}_{\mathrm{I}}^C = \emptyset \quad \Rightarrow \quad \bar{u}_{\mathrm{I}}^B = \emptyset$$

$$\mathcal{S}_N = \sum r(\bar{u}_{\mathrm{II}}^B)r(\bar{u}_{\mathrm{I}}^C)K_n(\bar{u}_{\mathrm{I}}^B|\bar{u}_{\mathrm{I}}^C)K_{N-n}(\bar{u}_{\mathrm{II}}^C|\bar{u}_{\mathrm{II}}^B)f(\bar{u}_{\mathrm{I}}^B, \bar{u}_{\mathrm{II}}^B)f(\bar{u}_{\mathrm{II}}^C, \bar{u}_{\mathrm{I}}^C)$$

$$r(u_k^C) = 0$$

Determinant formula

$$\mathcal{S}_N = \Delta_N(\bar{u}^C)\Delta'_N(\bar{u}^B)\det_N\Big[t(u_k^C, u_j^B)h(u_k^C, \bar{u}^B)\Big]$$

Izergin–Korepin formula

$$\bar{u}_{\mathrm{I}}^C = \emptyset \quad \Rightarrow \quad \bar{u}_{\mathrm{I}}^B = \emptyset$$

$$\mathcal{S}_N = \quad r(\bar{u}^B)K_N(\bar{u}^C|\bar{u}^B)$$

$$r(u_k^C) = 0$$

DWPF is the particular case of the scalar product.

The existence of a determinant representation for the scalar product implies the existence of a determinant representation for DWPF.

Scalar product in SU(3)-invariant models

Reshetikhin's representation

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^B_{\mathrm{I}}) r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^B_{\mathrm{I}}) r_3(\bar{v}^C_{\mathrm{II}}) f(\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}) f(\bar{u}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}}) f(\bar{v}^C_{\mathrm{II}}, \bar{v}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}})$$

$$\times \frac{f(\bar{v}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}})}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}^C_{\mathrm{II}}; \bar{u}^B_{\mathrm{II}} | \bar{v}^C_{\mathrm{I}}; \bar{v}^B_{\mathrm{I}}) Z_{k,b-n}(\bar{u}^B_{\mathrm{I}}; \bar{u}^C_{\mathrm{I}} | \bar{v}^B_{\mathrm{II}}; \bar{v}^C_{\mathrm{II}})$$

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$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^B_{\mathrm{I}}) r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^B_{\mathrm{I}}) r_3(\bar{v}^C_{\mathrm{II}}) f(\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}) f(\bar{u}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}}) f(\bar{v}^C_{\mathrm{II}}, \bar{v}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}})$$

$$\times \frac{f(\bar{v}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}})}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}^C_{\mathrm{II}}; \bar{u}^B_{\mathrm{II}} | \bar{v}^C_{\mathrm{I}}; \bar{v}^B_{\mathrm{I}}) Z_{k,b-n}(\bar{u}^B_{\mathrm{I}}; \bar{u}^C_{\mathrm{I}} | \bar{v}^B_{\mathrm{II}}; \bar{v}^C_{\mathrm{II}})$$

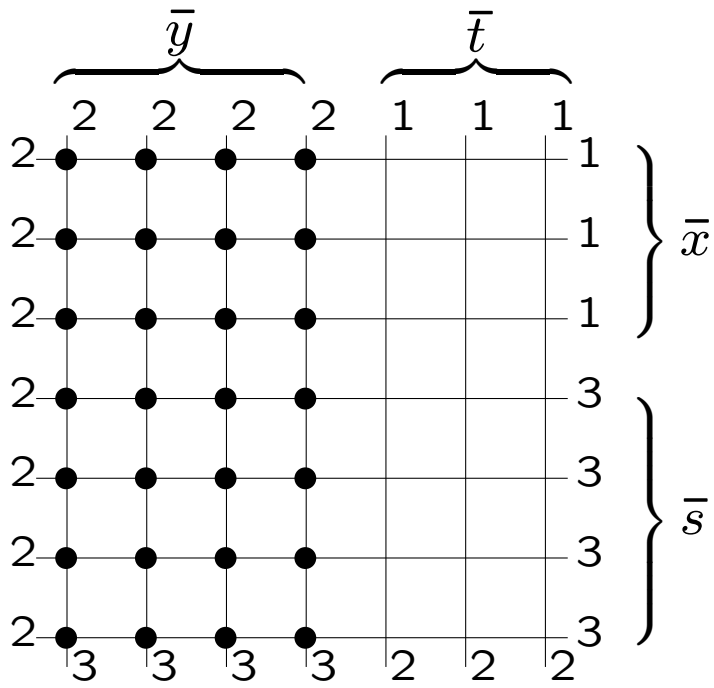
The sum is taken over partitions:

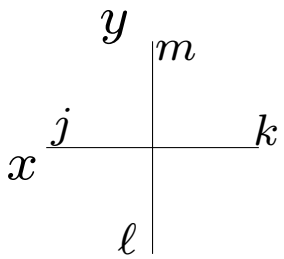
$$\bar{u}^B = \{\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}}\} \qquad \bar{v}^B = \{\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}}\} \qquad \#\bar{v}^B_{\mathrm{I}} = \#\bar{v}^C_{\mathrm{I}} = n = 0, 1, \ldots, b$$

$$\bar{u}^C = \{\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}\} \qquad \bar{v}^C = \{\bar{v}^C_{\mathrm{I}}, \bar{v}^C_{\mathrm{II}}\} \qquad \#\bar{u}^C_{\mathrm{I}} = \#\bar{u}^B_{\mathrm{I}} = k = 0, 1, \ldots, a$$

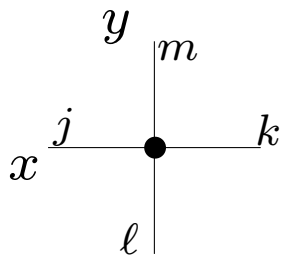
Reshetikhin Partition Function (RPF)

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) \quad =$$





$$= \Big(R(x,y)\Big)_{jk,\ell m}$$



$$= \Big(R^{t_1}(y,x)\Big)_{jk,\ell m}$$

Scalar product in SU(3)-invariant models

Reshetikhin's representation

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^B_{\mathrm{I}}) r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^B_{\mathrm{I}}) r_3(\bar{v}^C_{\mathrm{II}}) f(\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}) f(\bar{u}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}}) f(\bar{v}^C_{\mathrm{II}}, \bar{v}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}})$$

$$\times \frac{f(\bar{v}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}})}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}^C_{\mathrm{II}}; \bar{u}^B_{\mathrm{II}} | \bar{v}^C_{\mathrm{I}}; \bar{v}^B_{\mathrm{I}}) Z_{k,b-n}(\bar{u}^B_{\mathrm{I}}; \bar{u}^C_{\mathrm{I}} | \bar{v}^B_{\mathrm{II}}; \bar{v}^C_{\mathrm{II}})$$

Scalar product in SU(3)-invariant models

Reshetikhin's representation

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^B_{\mathrm{I}}) r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^B_{\mathrm{I}}) r_3(\bar{v}^C_{\mathrm{II}}) f(\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}) f(\bar{u}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}}) f(\bar{v}^C_{\mathrm{II}}, \bar{v}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}})$$

$$\times \frac{f(\bar{v}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}})}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}^C_{\mathrm{II}}; \bar{u}^B_{\mathrm{II}} | \bar{v}^C_{\mathrm{I}}; \bar{v}^B_{\mathrm{I}}) Z_{k,b-n}(\bar{u}^B_{\mathrm{I}}; \bar{u}^C_{\mathrm{I}} | \bar{v}^B_{\mathrm{II}}; \bar{v}^C_{\mathrm{II}})$$

Let $|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$ be on-shell Bethe vector

$$r_1(\bar{u}^B_{\mathrm{I}}) = \frac{f(\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}})}{f(\bar{u}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}})} f(\bar{v}^B, \bar{u}^B_{\mathrm{I}}), \qquad r_3(\bar{v}^B_{\mathrm{I}}) = \frac{f(\bar{v}^B_{\mathrm{II}}, \bar{v}^B_{\mathrm{I}})}{f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}})} f(\bar{v}^B_{\mathrm{I}}, \bar{u}^B)$$

Scalar product in SU(3)-invariant models

Reshetikhin's representation

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}_{\mathrm{I}}^B) r_1(\bar{u}_{\mathrm{II}}^C) r_3(\bar{v}_{\mathrm{I}}^B) r_3(\bar{v}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{I}}^C, \bar{u}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{II}}^B, \bar{u}_{\mathrm{I}}^B) f(\bar{v}_{\mathrm{II}}^C, \bar{v}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{I}}^B, \bar{v}_{\mathrm{II}}^B)$$

$$\times \frac{f(\bar{v}_{\mathrm{I}}^C, \bar{u}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{II}}^B, \bar{u}_{\mathrm{II}}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}_{\mathrm{II}}^C; \bar{u}_{\mathrm{II}}^B | \bar{v}_{\mathrm{I}}^C; \bar{v}_{\mathrm{I}}^B) Z_{k,b-n}(\bar{u}_{\mathrm{I}}^B; \bar{u}_{\mathrm{I}}^C | \bar{v}_{\mathrm{II}}^B; \bar{v}_{\mathrm{II}}^C)$$

Let $|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle$ be on-shell Bethe vector

$$r_1(\bar{u}_{\mathrm{I}}^B) = \frac{f(\bar{u}_{\mathrm{I}}^B, \bar{u}_{\mathrm{II}}^B)}{f(\bar{u}_{\mathrm{II}}^B, \bar{u}_{\mathrm{I}}^B)} f(\bar{v}^B, \bar{u}_{\mathrm{I}}^B), \qquad r_3(\bar{v}_{\mathrm{I}}^B) = \frac{f(\bar{v}_{\mathrm{II}}^B, \bar{v}_{\mathrm{I}}^B)}{f(\bar{v}_{\mathrm{I}}^B, \bar{v}_{\mathrm{II}}^B)} f(\bar{v}_{\mathrm{I}}^B, \bar{u}^B)$$

Special limit

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}_{\mathrm{I}}^B) r_1(\bar{u}_{\mathrm{II}}^C) r_3(\bar{v}_{\mathrm{I}}^B) r_3(\bar{v}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{I}}^C, \bar{u}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{II}}^B, \bar{u}_{\mathrm{I}}^B) f(\bar{v}_{\mathrm{II}}^C, \bar{v}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{I}}^B, \bar{v}_{\mathrm{II}}^B)$$

$$\times \frac{f(\bar{v}_{\mathrm{I}}^C, \bar{u}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{II}}^B, \bar{u}_{\mathrm{II}}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}_{\mathrm{II}}^C; \bar{u}_{\mathrm{II}}^B | \bar{v}_{\mathrm{I}}^C; \bar{v}_{\mathrm{I}}^B) Z_{k,b-n}(\bar{u}_{\mathrm{I}}^B; \bar{u}_{\mathrm{I}}^C | \bar{v}_{\mathrm{II}}^B; \bar{v}_{\mathrm{II}}^C)$$

Let $r_1(u_k^C) \to \infty$ and $r_3(v_k^C) = 0$. Then

$$\bar{u}_{\mathrm{I}}^C = \bar{u}_{\mathrm{I}}^B = \emptyset, \qquad \bar{v}_{\mathrm{II}}^C = \bar{v}_{\mathrm{II}}^B = \emptyset$$

Special limit

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^B_{\mathrm{I}}) r_1(\bar{u}^C_{\mathrm{II}}) r_3(\bar{v}^B_{\mathrm{I}}) r_3(\bar{v}^C_{\mathrm{II}}) f(\bar{u}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}}) f(\bar{u}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}}) f(\bar{v}^C_{\mathrm{II}}, \bar{v}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}})$$

$$\times \frac{f(\bar{v}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{I}}) f(\bar{v}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{II}})}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}^C_{\mathrm{II}}; \bar{u}^B_{\mathrm{II}} | \bar{v}^C_{\mathrm{I}}; \bar{v}^B_{\mathrm{I}}) Z_{k,b-n}(\bar{u}^B_{\mathrm{I}}; \bar{u}^C_{\mathrm{I}} | \bar{v}^B_{\mathrm{II}}; \bar{v}^C_{\mathrm{II}})$$

Let $r_1(u^C_k) \to \infty$ and $r_3(v^C_k) = 0$. Then

$$\bar{u}^C_{\mathrm{I}} = \bar{u}^B_{\mathrm{I}} = \emptyset, \qquad \bar{v}^C_{\mathrm{II}} = \bar{v}^B_{\mathrm{II}} = \emptyset$$

$$\mathcal{S}_{a,b} \to \frac{r_1(\bar{u}^C) r_3(\bar{v}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a,b}(\bar{u}^C; \bar{u}^B | \bar{v}^C; \bar{v}^B)$$

Special limit

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^B_{\rm I}) r_1(\bar{u}^C_{\rm II}) r_3(\bar{v}^B_{\rm I}) r_3(\bar{v}^C_{\rm II}) f(\bar{u}^C_{\rm I}, \bar{u}^C_{\rm II}) f(\bar{u}^B_{\rm II}, \bar{u}^B_{\rm I}) f(\bar{v}^C_{\rm II}, \bar{v}^C_{\rm I}) f(\bar{v}^B_{\rm I}, \bar{v}^B_{\rm II})$$

$$\times \frac{f(\bar{v}^C_{\rm I}, \bar{u}^C_{\rm I}) f(\bar{v}^B_{\rm II}, \bar{u}^B_{\rm II})}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} \, Z_{a-k,n}(\bar{u}^C_{\rm II}; \bar{u}^B_{\rm II} | \bar{v}^C_{\rm I}; \bar{v}^B_{\rm I}) Z_{k,b-n}(\bar{u}^B_{\rm I}; \bar{u}^C_{\rm I} | \bar{v}^B_{\rm II}; \bar{v}^C_{\rm II})$$

The coefficient $Z_{a,b}(\bar{u}^C; \bar{u}^B | \bar{v}^C; \bar{v}^B)$ is the particular case of the scalar product.

Representations for RFP

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = (-1)^b \sum K_b(\bar{s}-c|\bar{w}_{\mathrm{I}})K_a(\bar{w}_{\mathrm{II}}|\bar{t})K_b(\bar{y}|\bar{w}_{\mathrm{I}})f(\bar{w}_{\mathrm{I}},\bar{w}_{\mathrm{II}})$$

Here $\bar{w} = \{\bar{s},\ \bar{x}\}$. The sum is taken with respect to partitions of the set $\bar{w}$ into subsets $\bar{w}_{\mathrm{I}}$ and $\bar{w}_{\mathrm{II}}$ with $\#\bar{w}_{\mathrm{I}} = b$ and $\#\bar{w}_{\mathrm{II}} = a$.

Representations for RFP

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = (-1)^a f(\bar{y},\bar{x}) f(\bar{s},\bar{t}) \sum K_a(\bar{t}-c|\bar{\eta}_{\mathrm{I}}) K_a(\bar{x}|\bar{\eta}_{\mathrm{I}}) K_b(\bar{\eta}_{\mathrm{II}}-c|\bar{s}) f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})$$

Here $\bar{\eta} = \{\bar{y}+c,\ \bar{t}\}$. The sum is taken with respect to partitions of the set $\bar{\eta}$ into subsets $\bar{\eta}_{\mathrm{I}}$ and $\bar{\eta}_{\mathrm{II}}$ with $\#\bar{\eta}_{\mathrm{I}} = a$ and $\#\bar{\eta}_{\mathrm{II}} = b$.

Representations for RFP

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = \sum(-1)^n f(\bar{s},\bar{t}_{\rm I})f(\bar{y},\bar{x}_{\rm II})f(\bar{t}_{\rm I},\bar{t}_{\rm II})f(\bar{x}_{\rm II},\bar{x}_{\rm I})$$

$$\times K_n(\bar{x}_{\rm I}|\bar{t}_{\rm I})K_{a-n}(\bar{x}_{\rm II}|\bar{t}_{\rm II}-c)K_{b+n}(\bar{y},\bar{t}_{\rm I}-c|\bar{s},\bar{x}_{\rm I})$$

The sum is taken with respect to all partitions of the set $\bar{t}$ into subsets $\bar{t}_{\rm I}$, $\bar{t}_{\rm II}$ and the set $\bar{x}$ into subsets $\bar{x}_{\rm I}$, $\bar{x}_{\rm II}$ with $\#\bar{t}_{\rm I} = \#\bar{x}_{\rm I} = n$, $n = 0, 1, \ldots, a$.

Representations for RFP

Conjecture

There is no single determinant representation for $Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y})$.

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Corollary

There is no single determinant representation for the scalar products of on-shell Bethe vectors with arbitrary Bethe vectors.

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Corollary

There is no single determinant representation for the scalar products of on-shell Bethe vectors with arbitrary Bethe vectors.

We can consider special scalar products involving on-shell Bethe vector and some specific off-shell Bethe vector. In particular the scalar product of on-shell vector and twisted on-shell vector.

$$\mathcal{S}^{(\kappa)}_{a,b} \equiv \mathcal{S}^{(\kappa)}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi^{(\kappa)}_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$r_1(u^C_k) = \kappa \, \frac{f(u^C_k, \bar{u}^C_k)}{f(\bar{u}^C_k, u^C_k)} f(\bar{v}^C, u^C_k), \qquad r_3(v^C_k) = \kappa \, \frac{f(\bar{v}_k, v^C_k)}{f(v^C_k, \bar{v}_k)} f(v^C_k, \bar{u})$$

$$r_1(u^B_k) = \frac{f(u^B_k, \bar{u}^B_k)}{f(\bar{u}^B_k, u^B_k)} f(\bar{v}^B, u^B_k), \qquad r_3(v^B_k) = \frac{f(\bar{v}^B_k, v^B_k)}{f(v^B_k, \bar{v}^B_k)} f(v^B_k, \bar{u}^B)$$

The SU(2) analog of this scalar product is the generating function of σ^z-form factors in XXX SU(2)-invariant Heisenberg chain

$$Q_\kappa(m) = \frac{\tau_\kappa^m(0|\bar{u}^C)}{\tau^m(0|\bar{u}^B)} \langle \psi_n^{(\kappa)}(\bar{u}^C)|\psi_n(\bar{u}^B)\rangle$$

$$\langle \psi_n(\bar{u}^C)| \left(1 - \sigma_m^z\right) |\psi_n(\bar{u}^B)\rangle = 2\,\frac{d}{d\kappa}\Big(Q_\kappa(m) - Q_\kappa(m-1)\Big)\Big|_{\kappa=1}$$

The SU(2) analog of this scalar product is the generating function of σ^z-form factors in XXX SU(2)-invariant Heisenberg chain

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Similar formula exists for XXX SU(3)-invariant Heisenberg chain

$$Q_\kappa(m) = \frac{\tau_\kappa^m(0|\bar{u}^C;\bar{v}^C)}{\tau^m(0|\bar{u}^B;\bar{v}^B)} \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C;\bar{v}^C)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

$$\langle \psi_n(\bar{u}^C)|E_m^{2,2}|\psi_n(\bar{u}^B)\rangle = \frac{d}{d\kappa}\Big(Q_\kappa(m) - Q_\kappa(m-1)\Big)\Big|_{\kappa=1}$$

where elementary units $E_m^{\epsilon,\epsilon'}$ are: $\left(E^{\epsilon,\epsilon'}\right)_{jk} = \delta_{j\epsilon}\delta_{k\epsilon'}$

Determinant representation for SU(2) scalar product

$$\mathcal{S}_n^{(\kappa)}(\bar{u}^C;\bar{u}^B) = \langle \psi_n^{(\kappa)}(\bar{u}^C)|\psi_n(\bar{u}^B)\rangle = \Delta_n'(\bar{u}^C)\Delta_n(\bar{u}^B)\ \det_n\ \mathcal{M}$$

$$= \Delta_n'(\bar{u}^C)\Delta_n(\bar{u}^B)\ \det_n\ \mathcal{M}_\kappa$$

$\mathcal{M}$ and $\mathcal{M}_\kappa$ are matrices of the size $n \times n$

$$\mathcal{M}_\kappa = cg^{-1}(u_k^C, \bar{u}^B)\frac{\partial \tau_\kappa(u_k^B)}{\partial u_j^C} \qquad \mathcal{M} = cg^{-1}(u_j^B, \bar{u}^C)\frac{\partial \tau(u_j^C)}{\partial u_k^B}$$

Determinant representation for SU(2) scalar product

$$\mathcal{S}_n^{(\kappa)}(\bar{u}^C;\bar{u}^B) = \langle \psi_n^{(\kappa)}(\bar{u}^C)|\psi_n(\bar{u}^B)\rangle = \Delta'_n(\bar{u}^C)\Delta_n(\bar{u}^B)\ \det_n\ \mathcal{M}$$

$$= \Delta'_n(\bar{u}^C)\Delta_n(\bar{u}^B)\ \det_n\ \mathcal{M}_\kappa$$

$\mathcal{M}$ and $\mathcal{M}_\kappa$ are matrices of the size $n \times n$

$$\mathcal{M}_\kappa = cg^{-1}(u_k^C, \bar{u}^B)\frac{\partial \tau_\kappa(u_k^B)}{\partial u_j^C} \qquad \mathcal{M} = cg^{-1}(u_j^B, \bar{u}^C)\frac{\partial \tau(u_j^C)}{\partial u_k^B}$$

Determinant representation for SU(2) scalar product

$$\mathcal{S}_n^{(\kappa)}(\bar{u}^C;\bar{u}^B) = \langle \psi_n^{(\kappa)}(\bar{u}^C)|\psi_n(\bar{u}^B)\rangle = \Delta_n'(\bar{u}^C)\Delta_n(\bar{u}^B)\ \det_n\ \mathcal{M}$$

$$= \Delta_n'(\bar{u}^C)\Delta_n(\bar{u}^B)\ \det_n\ \mathcal{M}_\kappa$$

$\mathcal{M}$ and $\mathcal{M}_\kappa$ are matrices of the size $n \times n$

$$\mathcal{M}_\kappa = cg^{-1}(u_k^C, \bar{u}^B)\frac{\partial \tau_\kappa(u_k^B)}{\partial u_j^C} \qquad \mathcal{M} = cg^{-1}(u_j^B, \bar{u}^C)\frac{\partial \tau(u_j^C)}{\partial u_k^B}$$

$$\tau_\kappa(w|\bar{u}^C) = r(w)f(\bar{u}^C, w) + \kappa\, f(w, \bar{u}^C)$$

$$\tau(w|\bar{u}^B) = r(w)f(\bar{u}^B, w) + f(w, \bar{u}^B)$$

Scalar product of on-shell vector and twisted on-shell vector (SU(3))

$$\mathcal{S}_{a,b}^{(\kappa)} \equiv \mathcal{S}_{a,b}^{(\kappa)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

Scalar product of on-shell vector and twisted on-shell vector (SU(3))

$$\mathcal{S}_{a,b}^{(\kappa)} \equiv \mathcal{S}_{a,b}^{(\kappa)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b}^{(\kappa)} = \sum r_1(\bar{u}_{\mathrm{I}}^B) r_1(\bar{u}_{\mathrm{II}}^C) r_3(\bar{v}_{\mathrm{I}}^B) r_3(\bar{v}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{I}}^C, \bar{u}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{II}}^B, \bar{u}_{\mathrm{I}}^B) f(\bar{v}_{\mathrm{II}}^C, \bar{v}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{I}}^B, \bar{v}_{\mathrm{II}}^B)$$

$$\times \frac{f(\bar{v}_{\mathrm{I}}^C, \bar{u}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{II}}^B, \bar{u}_{\mathrm{II}}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}_{\mathrm{II}}^C; \bar{u}_{\mathrm{II}}^B | \bar{v}_{\mathrm{I}}^C; \bar{v}_{\mathrm{I}}^B) Z_{k,b-n}(\bar{u}_{\mathrm{I}}^B; \bar{u}_{\mathrm{I}}^C | \bar{v}_{\mathrm{II}}^B; \bar{v}_{\mathrm{II}}^C)$$

Scalar product of on-shell vector and twisted on-shell vector (SU(3))

$$\mathcal{S}_{a,b}^{(\kappa)} \equiv \mathcal{S}_{a,b}^{(\kappa)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

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$$\times \frac{f(\bar{v}_{\mathrm{I}}^C, \bar{u}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{II}}^B, \bar{u}_{\mathrm{II}}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}_{\mathrm{II}}^C; \bar{u}_{\mathrm{II}}^B | \bar{v}_{\mathrm{I}}^C; \bar{v}_{\mathrm{I}}^B) Z_{k,b-n}(\bar{u}_{\mathrm{I}}^B; \bar{u}_{\mathrm{I}}^C | \bar{v}_{\mathrm{II}}^B; \bar{v}_{\mathrm{II}}^C)$$

$$r_1(u_k^C) = \kappa \frac{f(u_k^C, \bar{u}_k^C)}{f(\bar{u}_k^C, u_k^C)} f(\bar{v}^C, u_k^C), \qquad r_3(v_k^C) = \kappa \frac{f(\bar{v}_k, v_k^C)}{f(v_k^C, \bar{v}_k)} f(v_k^C, \bar{u})$$

$$r_1(u_k^B) = \frac{f(u_k^B, \bar{u}_k^B)}{f(\bar{u}_k^B, u_k^B)} f(\bar{v}^B, u_k^B), \qquad r_3(v_k^B) = \frac{f(\bar{v}_k^B, v_k^B)}{f(v_k^B, \bar{v}_k^B)} f(v_k^B, \bar{u}^B)$$

Scalar product of on-shell vector and twisted on-shell vector (SU(3))

$$\mathcal{S}_{a,b}^{(\kappa)} \equiv \mathcal{S}_{a,b}^{(\kappa)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\mathcal{S}_{a,b}^{(\kappa)} = \sum r_1(\bar{u}_{\mathrm{I}}^B) r_1(\bar{u}_{\mathrm{II}}^C) r_3(\bar{v}_{\mathrm{I}}^B) r_3(\bar{v}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{I}}^C, \bar{u}_{\mathrm{II}}^C) f(\bar{u}_{\mathrm{II}}^B, \bar{u}_{\mathrm{I}}^B) f(\bar{v}_{\mathrm{II}}^C, \bar{v}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{I}}^B, \bar{v}_{\mathrm{II}}^B)$$

$$\times \frac{f(\bar{v}_{\mathrm{I}}^C, \bar{u}_{\mathrm{I}}^C) f(\bar{v}_{\mathrm{II}}^B, \bar{u}_{\mathrm{II}}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}_{\mathrm{II}}^C; \bar{u}_{\mathrm{II}}^B | \bar{v}_{\mathrm{I}}^C; \bar{v}_{\mathrm{I}}^B) Z_{k,b-n}(\bar{u}_{\mathrm{I}}^B; \bar{u}_{\mathrm{I}}^C | \bar{v}_{\mathrm{II}}^B; \bar{v}_{\mathrm{II}}^C)$$

$$r_1(u_k^C) = \kappa \frac{f(u_k^C, \bar{u}_k^C)}{f(\bar{u}_k^C, u_k^C)} f(\bar{v}^C, u_k^C), \qquad r_3(v_k^C) = \kappa \frac{f(\bar{v}_k, v_k^C)}{f(v_k^C, \bar{v}_k)} f(v_k^C, \bar{u})$$

$$r_1(u_k^B) = \frac{f(u_k^B, \bar{u}_k^B)}{f(\bar{u}_k^B, u_k^B)} f(\bar{v}^B, u_k^B), \qquad r_3(v_k^B) = \frac{f(\bar{v}_k^B, v_k^B)}{f(v_k^B, \bar{v}_k^B)} f(v_k^B, \bar{u}^B)$$

Determinant representation for SU(3) scalar product

$$\mathcal{S}_{a,b}^{(\kappa)} = t(\bar{v}^C, \bar{u}^B)\Delta'_a(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^C)\Delta_b(\bar{v}^B) \det_{a+b} \mathcal{N}$$

$\mathcal{N}$ is a block-matrix of the size $(a+b) \times (a+b)$

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}^{(u)}(u_j^C, u_k^B) & \mathcal{N}^{(u)}(u_j^C, v_k^C) \\ \\ \mathcal{N}^{(v)}(v_j^B, u_k^B) & \mathcal{N}^{(v)}(v_j^B, v_k^C) \end{pmatrix}$$

Determinant representation for SU(3) scalar product

$$\mathcal{S}_{a,b}^{(\kappa)} = t(\bar{v}^C, \bar{u}^B)\Delta'_a(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^C)\Delta_b(\bar{v}^B) \det_{a+b} \mathcal{N}$$

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$$\mathcal{N}^{(u)}(u_j^C, w_k) = cg^{-1}(w_k, \bar{u}^C)g^{-1}(\bar{v}^C, w_k)\frac{\partial \tau_\kappa(w_k)}{\partial u_j^C}, \quad w_k = u_k^B,\ v_k^C$$

$$\mathcal{N}^{(v)}(v_j^B, w_k) = -cg^{-1}(\bar{v}^B, w_k)g^{-1}(w_k, \bar{u}^B)\frac{\partial \tau(w_k)}{\partial v_j^B}, \quad w_k = u_k^B,\ v_k^C$$

Determinant representation for SU(3) scalar product

$$\mathcal{S}^{(\kappa)}_{a,b} = t(\bar{v}^C, \bar{u}^B)\Delta'_a(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^C)\Delta_b(\bar{v}^B)\det_{a+b}\mathcal{N}$$

$\mathcal{N}$ is a block-matrix of the size $(a+b)\times(a+b)$

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}^{(u)}(u^C_j, u^B_k) & \mathcal{N}^{(u)}(u^C_j, v^C_k) \\ \\ \mathcal{N}^{(v)}(v^B_j, u^B_k) & \mathcal{N}^{(v)}(v^B_j, v^C_k) \end{pmatrix}$$

$$\mathcal{N}^{(u)}(u^C_j, w_k) = cg^{-1}(w_k, \bar{u}^C)g^{-1}(\bar{v}^C, w_k)\frac{\partial \tau_\kappa(w_k)}{\partial u^C_j}, \quad w_k = u^B_k,\ v^C_k$$

$$\mathcal{N}^{(v)}(v^B_j, w_k) = -cg^{-1}(\bar{v}^B, w_k)g^{-1}(w_k, \bar{u}^B)\frac{\partial \tau(w_k)}{\partial v^B_j}, \quad w_k = u^B_k,\ v^C_k$$

Perspectives

$$\langle \psi_{a,b}^{(\kappa)}(\bar{u}^C;\bar{v}^C)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle \longrightarrow \langle \psi_{a,b}(\bar{u}^C;\bar{v}^C)|E_m^{2,2}|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

Perspectives

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Is it possible to calculate other form factors?

$$\mathcal{F}_m^{\epsilon,\epsilon'} = \langle \psi_{a',b'}(\bar{u}^C;\bar{v}^C)|E_m^{\epsilon,\epsilon'}|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle \quad = ?$$

Perspectives

$$\langle \psi_{a,b}^{(\kappa)}(\bar{u}^C;\bar{v}^C)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle \longrightarrow \langle \psi_{a,b}(\bar{u}^C;\bar{v}^C)|E_m^{2,2}|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

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Inverse scattering problem

$$E_m^{\epsilon,\epsilon'} = \mathcal{T}^{m-1}(0)\, T_{\epsilon',\epsilon}(0)\, \mathcal{T}^{-m}(0)$$

$$\mathcal{F}_m^{\epsilon,\epsilon'} = \frac{\tau^{m-1}(0|\bar{u}^C,\bar{v}^C)}{\tau^m(0|\bar{u}^B,\bar{v}^B)}\langle \psi_{a',b'}(\bar{u}^C;\bar{v}^C)|T_{\epsilon',\epsilon}(0)|\psi_{a,b}(\bar{u}^B;\bar{v}^B)\rangle$$

Perspectives

$$\mathcal{F}_m^{2,2} \quad \sim \quad \frac{d}{d\kappa} \det_{a+b} \begin{pmatrix} (*)\dfrac{\partial \tau_\kappa}{\partial u_j^C} \\ \hline (*)\dfrac{\partial \tau}{\partial v_j^B} \end{pmatrix}_{\kappa=1}$$

Perspectives

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After setting $\kappa = 1$ the eigenvalue τ_κ turns into τ

$$\mathcal{F}_m^{2,2} \quad \sim \quad \det_{a+b} \begin{pmatrix} (*)\dfrac{\partial \tau}{\partial s_j} \\ \\ \text{modified row} \end{pmatrix} \qquad s_j = u_j^C,\ v_j^B$$

Perspectives

$$\mathcal{F}_m^{2,2} \quad \sim \quad \frac{d}{d\kappa} \det_{a+b} \begin{pmatrix} (*)\dfrac{\partial \tau_\kappa}{\partial u_j^C} \\ \hline (*)\dfrac{\partial \tau}{\partial v_j^B} \end{pmatrix}_{\kappa=1}$$

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Perspectives

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Perspectives

We should calculate the action of $T_{ij}(w)$ on Bethe vectors

$$T_{ij}(w)|\psi_{a,b}(\bar{u};\bar{v})\rangle = |\phi(w|\bar{u};\bar{v})\rangle$$

Is it possible to present the result of this action as a linear combination of off-shell Bethe vectors?

$$|\phi(w|\bar{u};\bar{v})\rangle = \sum_k \alpha_k \, |\psi_{a',b'}(\bar{u}^{(k)};\bar{v}^{(k)})\rangle$$

Perspectives

We should calculate the action of $T_{ij}(w)$ on Bethe vectors

$$T_{ij}(w)|\psi_{a,b}(\bar{u};\bar{v})\rangle = |\phi(w|\bar{u};\bar{v})\rangle$$

Is it possible to present the result of this action as a linear combination of off-shell Bethe vectors?

$$|\phi(w|\bar{u};\bar{v})\rangle = \sum_k \alpha_k \, |\psi_{a',b'}(\bar{u}^{(k)};\bar{v}^{(k)})\rangle$$

$$|\psi_{a,b}(\bar{u};\bar{v})\rangle = P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big)|0\rangle$$

It is not *apriori* obvious that $|\psi_{a,b}(\bar{u};\bar{v})\rangle$ can be presented as a linear combination of the polynomials of the same type.

Summation over partitions

- We use different representations for RPF $Z_{a-k,n}(\bar{u}^C_{\mathrm{II}};\bar{u}^B_{\mathrm{II}}|\bar{v}^C_{\mathrm{I}};\bar{v}^B_{\mathrm{I}})$ and $Z_{k,b-n}(\bar{u}^B_{\mathrm{I}};\bar{u}^C_{\mathrm{I}}|\bar{v}^B_{\mathrm{II}};\bar{v}^C_{\mathrm{II}})$.

- We use summation identities for DWPF $K_n(\bar{x}|\bar{y})$

$$\sum K_{m_1}(\bar{z}_{\mathrm{I}}|\bar{x})K_{m_2}(\bar{y}|\bar{z}_{\mathrm{II}})f(\bar{z}_{\mathrm{II}},\bar{z}_{\mathrm{I}}) = (-1)^{m_1}f(\bar{z},\bar{x})K_{m_1+m_2}(\bar{x}-c,\bar{y}|\bar{z})$$

The sum is taken over partitions of the set $\bar{z}$ into subsets $\bar{z}_{\mathrm{I}}$ and $\bar{z}_{\mathrm{II}}$ with $\#\bar{z}_{\mathrm{I}} = m_1$ and $\#\bar{z}_{\mathrm{II}} = m_2$.

Summation over partitions

- We use different representations for RPF $Z_{a-k,n}(\bar{u}^C_{\mathrm{II}};\bar{u}^B_{\mathrm{II}}|\bar{v}^C_{\mathrm{I}};\bar{v}^B_{\mathrm{I}})$ and $Z_{k,b-n}(\bar{u}^B_{\mathrm{I}};\bar{u}^C_{\mathrm{I}}|\bar{v}^B_{\mathrm{II}};\bar{v}^C_{\mathrm{II}})$.

- We use summation identities for DWPF $K_n(\bar{x}|\bar{y})$

$$\sum f(\bar{y}_{\mathrm{II}},\bar{y}_{\mathrm{I}})f(\bar{x}_{\mathrm{I}},\bar{x}_{\mathrm{II}})K_{m_{\mathrm{I}}}(\bar{y}_{\mathrm{I}}|\bar{x}_{\mathrm{I}})K_{m_{\mathrm{II}}}(\bar{x}_{\mathrm{II}}+c|\bar{y}_{\mathrm{II}}) = (-1)^m t(\bar{x},\bar{y})h(\bar{x},\bar{x})h(\bar{y},\bar{y})$$

The sum is taken over all possible partitions of the sets $\bar{x}$ and $\bar{y}$ with $\#\bar{x}_{\mathrm{I}} = \#\bar{y}_{\mathrm{I}} = m_{\mathrm{I}}$ and $\#\bar{x}_{\mathrm{II}} = \#\bar{y}_{\mathrm{II}} = m_{\mathrm{II}}$.

Conjecture

$$\mathcal{S}_{a,b} = \Delta'_a(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^C)\Delta_b(\bar{v}^B)\det_{a+b}\mathcal{N}$$

$\mathcal{N}$ is a block-matrix of the size $(a+b)\times(a+b)$

$$\mathcal{N} = \begin{pmatrix} (*)\dfrac{\partial\tau(u_k^C)}{\partial u_j^B} & (*)\dfrac{\partial\tau(v_k^C)}{\partial u_j^B} \\ \\ (*)\dfrac{\partial\tau(u_k^C)}{\partial v_j^B} & (*)\dfrac{\partial\tau(v_k^C)}{\partial v_j^B} \end{pmatrix}$$

Conjecture

$$\mathcal{S}_{a,b} = \Delta'_a(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^C)\Delta_b(\bar{v}^B) \det_{a+b} \mathcal{N}$$

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Simple tests for small a and b do not confirm this conjecture.