Algebraic Bethe Ansatz for scalar products
in SU(3)-invariant models

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Algebraic Bethe Ansatz for scalar products in SU(3)-invariant models

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**Problem**

Suppose that we have a quantum model and we want to calculate matrix elements of some operator $\hat{O}$

$$O_{\psi,\psi'} = \langle \psi | \hat{O} | \psi' \rangle$$

where $|\psi\rangle$ and $|\psi'\rangle$ are eigenstates of the Hamiltonian.
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where $|\psi\rangle$ and $|\psi'\rangle$ are eigenstates of the Hamiltonian.

Suppose that the action of $\hat{O}$ to the right or to the left is known

$$\hat{O} |\psi'\rangle = |\phi'\rangle \quad \langle \psi | \hat{O} = \langle \phi |$$

Then we reduce the problem to the calculation of the scalar product, where one of the states is the eigenstate of the Hamiltonian.

$$O_{\psi, \psi'} = \langle \psi | \phi' \rangle \quad O_{\psi, \psi'} = \langle \phi | \psi' \rangle$$
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\( \mathfrak{gl}_{2} \)-based models (rational and trigonometric \( R \)-matrix)

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- **Quantum Inverse Scattering Problem**

N. Kitanine, J.M. Maillet, V. Terras '99

J.M. Maillet, V. Terras '00 (Higher rank algebras)

\[
\hat{O}|\psi\rangle = |\phi\rangle
\]
Higher rank algebras

- Nested Bethe ansatz

(P.P. Kulish, N.Yu. Reshetikhin, ’83)
Higher rank algebras

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- Other formulations of nested Bethe ansatz
  V.O. Tarasov, A.N. Varchenko ’95
  S. Belliard, S. Khoroshkin, S. Pakuliak, E. Ragoucy ’08, ’10
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- **Scalar products in SU(3)-invariant models**
  (N.Yu. Reshetikhin, ’86)
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  J. Escobedo, N. Gromov, A. Sever, P. Vieira ’11
  B. Pozsgay, W.-V. van G. Oei, M. Kormos ’12
  M. Wheeler ’12
Algebraic Bethe Ansatz

\[ R_{12}(u, v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u, v) \]
Algebraic Bethe Ansatz

\[ R_{12}(u, v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u, v) \]

Algebraic Bethe Ansatz works if there exists a pseudovacuum vector \( |0\rangle \) and dual pseudovacuum vector \( \langle 0| \)

\[ T_{jj}(u)|0\rangle = r_j(u)|0\rangle, \quad T_{jk}(u)|0\rangle = 0, \quad j > k \]

\[ \langle 0|T_{jj}(u) = r_j(u)\langle 0|, \quad \langle 0|T_{jk}(u) = 0, \quad j < k \]

One can set one of \( r_j(u) \) equals to 1 without loss of generality. Other \( r_j(u) \) remain free functional parameters (generalized model).
Bethe vectors

We look for the eigenvectors of the transfer matrix

$$\mathcal{T}(w) = \text{tr} \ T(w)$$
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$$\mathcal{T}(w) = \text{tr} \, T(w)$$

$$|\psi(\bar{u})\rangle = P(T_{ij}(u_k))|0\rangle, \quad i < j, \quad \bar{u} = u_1, \ldots, u_n$$

We say that $|\psi(\bar{u})\rangle$ is an off-shell Bethe vector, if the parameters $\bar{u}$ are generic complex numbers.
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$$\mathcal{T}(w) = \text{tr} \, T(w)$$

$$|\psi(\bar{u})\rangle = P\left(T_{i,j}(u_k)\right)|0\rangle, \quad i < j, \quad \bar{u} = u_1, \ldots, u_n$$

We say that $|\psi(\bar{u})\rangle$ is an off-shell Bethe vector, if the parameters $\bar{u}$ are generic complex numbers.

If the parameters $\bar{u}$ satisfy the system of Bethe equations, then we say that $|\psi(\bar{u})\rangle$ is an on-shell Bethe vector

$$\mathcal{T}(w)|\psi(\bar{u})\rangle = \tau(w|\bar{u})|\psi(\bar{u})\rangle$$
The rules of the game

What is given:

- The $R$-matrix (thus, the commutation relations $[T_{ij}(u), T_{k\ell}(v)]$).
- The action $T_{ij}(u)|0\rangle$ for $i \leq j$.
- The form of the Bethe vectors $|\psi(\bar{u})\rangle = P(T_{ij}(u_k))|0\rangle$, $i < j$ (that is the polynomials $P(T_{ij}(u_k))$ are fixed).
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What is given:

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What we want to find:

$$\langle \psi(\bar{u}^C) | \psi(\bar{u}^B) \rangle = \langle 0 | P(T_{k\ell}(u_k^C)) P(T_{ij}(u_k^B)) | 0 \rangle, \quad k > \ell, \quad i < j$$
We consider $R$-matrix of the form

$$R(u - v) = I + g(u, v)P, \quad g(u, v) = \frac{c}{u - v}$$

Other rational functions often appearing in the formulas

$$f(u, v) = 1 + g(u, v)$$

$$h(u, v) = \frac{f(u, v)}{g(u, v)}$$

$$t(u, v) = \frac{g(u, v)}{h(u, v)}$$
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$$

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$$

Other rational functions often appearing in the formulas

$$
f(u, v) = 1 + g(u, v) = \frac{u - v + c}{u - v}
$$

$$
h(u, v) = \frac{f(u, v)}{g(u, v)} = \frac{u - v + c}{c}
$$

$$
t(u, v) = \frac{g(u, v)}{h(u, v)} = \frac{c^2}{(u - v + c)(u - v)}
$$
Shorthand notations for products

\[ T_{ij}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{ij}(w_k) \]

\[ r_1(\bar{u}_II^C) = \prod_{u_k^C \in \bar{u}_II^C} r_1(u_k^C) \]

\[ h(\bar{u}_C^C, v_j) = \prod_{u_k^C \in \bar{u}_C^C} h(u_k^C, v_j) \]

\[ f(\bar{u}_I^B, \bar{u}_II^B) = \prod_{u_j^B \in \bar{u}_I^B} \prod_{u_k^B \in \bar{u}_II^B} f(u_j^B, u_k^B) \]
Shorthand notations for products

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\[ f(\bar{u}_I, \bar{u}_II) = \prod_{u_j^B \in \bar{u}_I} \prod_{u_k^B \in \bar{u}_II} f(u_j^B, u_k^B) \]
\[ \Delta_n(\bar{u}) = \prod_{j > k} g(u_j, u_k), \quad \Delta'_n(\bar{u}) = \prod_{j < k} g(u_j, u_k) \]
Shorthand notations for products

\[ T_{ij}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{ij}(w_k) \]
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\[ h(\bar{u}_C^C, v_j) = \prod_{u_k^C \in \bar{u}_C^C} h(u_k^C, v_j) \]
\[ f(\bar{u}_I^B, \bar{u}_II^B) = \prod_{u_j^B \in \bar{u}_I^B} \prod_{u_k^B \in \bar{u}_II^B} f(u_j^B, u_k^B) \]

Special subsets

\[ \bar{u}_j = \bar{u} \setminus u_j, \quad \bar{v}_k^C = \bar{v}^C \setminus v_k^C, \ldots \]
Algebraic Bethe Ansatz for SU(2) models

\[ T(w) = \left( \begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array} \right) \]

\[ A(w)|0\rangle = r(w)|0\rangle, \quad D(w)|0\rangle = |0\rangle, \quad C(w)|0\rangle = 0 \]

\[ \langle 0|A(w) = r(w)\langle 0|, \quad \langle 0|D(w) = \langle 0|, \quad \langle 0|B(w) = 0 \]

\( r(w) \) is a free functional parameter.
Algebraic Bethe Ansatz for SU(2) models

\[ T(w) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \]

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Bethe vectors

\[ |\psi_n(\bar{u})\rangle = B(\bar{u})|0\rangle, \quad \langle \psi_n(\bar{u})| = \langle 0|C(\bar{u}), \quad \bar{u} = u_1, \ldots, u_n \]
On-shell Bethe vectors

\[ |\psi_n(\bar{u})\rangle = B(\bar{u})|0\rangle, \quad \langle \psi_n(\bar{u})| = \langle 0|C(\bar{u}), \quad \bar{u} = u_1, \ldots, u_n \]

If the set \( \bar{u} \) satisfies Bethe equations

\[ r(u_k) = \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)}, \quad k = 1, \ldots, n \quad \bar{u}_k = \bar{u} \setminus u_k \]

then

\[ \mathcal{T}(w)|\psi_n(\bar{u})\rangle = \tau(w|\bar{u})|\psi_n(\bar{u})\rangle \]

\[ \tau(w|\bar{u}) = r(w)f(\bar{u}, w) + f(w, \bar{u}) \]
Algebraic Bethe Ansatz for SU(3) models

\[ T_{11}(w)|0\rangle = r_1(w)|0\rangle, \quad T_{22}(w)|0\rangle = |0\rangle, \quad T_{33}(w)|0\rangle = r_3(w)|0\rangle \]

\[ T_{jk}(w)|0\rangle = 0, \quad j > k; \quad \langle 0|T_{jk}(w) = 0, \quad j < k \]
Algebraic Bethe Ansatz for SU(3) models

\[ T_{11}(w)|0\rangle = r_1(w)|0\rangle, \quad T_{22}(w)|0\rangle = |0\rangle, \quad T_{33}(w)|0\rangle = r_3(w)|0\rangle \]

\[ T_{jk}(w)|0\rangle = 0, \quad j > k; \quad \langle 0|T_{jk}(w) = 0, \quad j < k \]

**Bethe vectors**

\[ |\psi_{a,b}(\bar{u}; \bar{v})\rangle = P\left(T_{ij}(u_k), T_{ij}(v_k)\right)|0\rangle, \quad i < j, \quad \bar{u} = u_1, \ldots, u_a \]

\[ \langle \psi_{a,b}(\bar{u}; \bar{v})| = \langle 0|P\left(T_{ij}(u_k), T_{ij}(v_k)\right), \quad i > j, \quad \bar{v} = v_1, \ldots, v_b \]

\[ |\psi_{1,1}(u; v)\rangle = T_{12}(u)T_{23}(v)|0\rangle + g(v, u)T_{13}(u)|0\rangle \]
On-shell Bethe vectors

\[ \mathcal{T}(w) | \psi_{a,b}(\bar{u}; \bar{v}) \rangle = \tau(w|\bar{u}, \bar{v}) | \psi_{a,b}(\bar{u}; \bar{v}) \rangle \]

\[ \tau(w|\bar{u}, \bar{v}) = r_1(w) f(\bar{u}, w) + f(w, \bar{u}) f(\bar{v}, w) + r_3(w) f(w, \bar{v}) \]

Bethe equations

\[ r_1(u_k) = \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)} f(\bar{v}, u_k), \quad r_3(v_k) = \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)} f(v_k, \bar{u}) \]

\[ \bar{u}_k = \bar{u} \setminus u_k, \quad \bar{v}_k = \bar{v} \setminus v_k \]
Twisted transfer matrix

\[ R_{12}(u, v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u, v) \]

Let \( \tilde{T}(w) = \tilde{\kappa}T(w) \), where \([\tilde{\kappa}_1\tilde{\kappa}_2, R_{12}] = 0\)
Twisted transfer matrix

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\[ R_{12}(u, v)\tilde{T}_1(u)\tilde{T}_2(v) = \tilde{T}_2(v)\tilde{T}_1(u)R_{12}(u, v) \]

\[ \tilde{\kappa} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\( \kappa \) is called twist parameter
Twisted transfer matrix

\[ R_{12}(u, v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u, v) \]

Let \( \tilde{T}(w) = \tilde{\kappa}T(w) \), where \([\tilde{\kappa}_1\tilde{\kappa}_2, R_{12}] = 0 \)

\[ R_{12}(u, v)\tilde{T}_1(u)\tilde{T}_2(v) = \tilde{T}_2(v)\tilde{T}_1(u)R_{12}(u, v) \]

Twisted on-shell vectors

\[ \mathcal{T}_\kappa(w) = \text{tr} \tilde{T}(w) \]

\[ \mathcal{T}_\kappa(w)\ket{\psi^{(\kappa)}_{a,b}(\bar{u}; \bar{v})} = \tau_\kappa(w)\ket{\psi^{(\kappa)}_{a,b}(\bar{u}; \bar{v})} \]

\[ \tau_\kappa(w) \equiv \tau_\kappa(w|\bar{u}, \bar{v}) = r_1(w)f(\bar{u}, w) + \kappa f(w, \bar{u})f(\bar{v}, w) + r_3(w)f(w, \bar{v}) \]
Twisted transfer matrix

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Let \( \tilde{T}(w) = \hat{\kappa}T(w) \), where \([\hat{\kappa}_1 \hat{\kappa}_2, R_{12}] = 0\)

\[ R_{12}(u, v)\tilde{T}_1(u)\tilde{T}_2(v) = \tilde{T}_2(v)\tilde{T}_1(u)R_{12}(u, v) \]

Twisted on-shell vectors

\[ T_\kappa(w) = \text{tr} \tilde{T}(w) \]
\[ T_\kappa(w)|\psi^{(\kappa)}_{a,b}(\bar{u}; \bar{v})\rangle = \tau_\kappa(w)|\psi^{(\kappa)}_{a,b}(\bar{u}; \bar{v})\rangle \]

\[ r_1(u_k) = \kappa \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)}f(\bar{v}, u_k), \quad r_3(v_k) = \kappa \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)}f(v_k, \bar{u}) \]
Scalar product in SU(2)-invariant models

Izergin–Korepin representation

\[ S_N \equiv S_N(\vec{u}^C; \vec{u}^B) = \langle \psi_n(\vec{u}^C) | \psi_n(\vec{u}^B) \rangle \]
Scalar product in SU(2)-invariant models

Izergin–Korepin representation

\[ S_N \equiv S_N(\vec{u}^C; \vec{u}^B) = \langle \psi_n(\vec{u}^C) | \psi_n(\vec{u}^B) \rangle \]

\[ S_N = \sum r(\vec{u}^B_I) r(\vec{u}^C_I) K_n(\vec{u}^B_I | \vec{u}^C_I) K_{N-n}(\vec{u}^C_{II} | \vec{u}^B_{II}) f(\vec{u}^C_I, \vec{u}^C_{II}) f(\vec{u}^B_I, \vec{u}^B_{II}) \]

The sum is taken over partitions of the sets \( \vec{u}^C \) and \( \vec{u}^B \)

\( \vec{u}^B = \{ \vec{u}^B_I, \vec{u}^B_{II} \} \)

\( \#\vec{u}^B = \#\vec{u}^C = n, \quad n = 0, 1, \ldots, N \)

\( \vec{u}^C = \{ \vec{u}^C_I, \vec{u}^C_{II} \} \)
Domain Wall Partition Function (DWPF)

$$K_n(\bar{x}|\bar{y}) = \ldots$$
Domain Wall Partition Function (DWPF)

Determinant representation (Izergin '87)

\[ K_n(\bar{x}|\bar{y}) = \Delta_n(\bar{x}) \Delta'_n(\bar{y}) h(\bar{x}, \bar{y}) \det_n t(x_j, y_k) \]
Domain Wall Partition Function (DWPF)

Determinant representation (Izergin '87)

\[ K_n(\bar{x}|\bar{y}) = \Delta_n(\bar{x}) \Delta'_n(\bar{y}) h(\bar{x}, \bar{y}) \det_n t(x_j, y_k) \]

\[ K_n(\bar{x}|\bar{y}) = \frac{\prod_{j,k=1}^{n} (x_j - y_k + c)}{\prod_{j>k}^{n} (x_j - x_k)(y_k - y_j)} \det_n \left[ \frac{c}{(x_j - y_k)(x_j - y_k + c)} \right] \]
Scalar product in SU(2)-invariant models

\[ S_N = \langle \psi_n(\bar{u}^C) | \psi_n(\bar{u}^B) \rangle \]

Suppose that \( |\psi_n(\bar{u}^B)\rangle \) is on-shell vector, i.e. the set \( \bar{u}^B \) satisfies Bethe equations
Scalar product in SU(2)-invariant models

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Suppose that \( |\psi_n(\bar{u}^B)\rangle \) is on-shell vector, i.e. the set \( \bar{u}^B \) satisfies Bethe equations

\[ S_N = \Delta_N(\bar{u}^C) \Delta'_N(\bar{u}^B) \det M_{jk} \]

\[ M_{jk} = t(u^C_k, u^B_j) h(u^C_k, \bar{u}^B) - (-1)^N r(u^C_k) t(u^B_j, u^C_k) h(\bar{u}^B, u^C_k) \]
Scalar product in SU(2)-invariant models

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It is easy to check that the entries \( M_{jk} \) are proportional to the Jacobian of the transfer matrix eigenvalue

\[ M_{jk} = cg^{-1}(u^C_k, \bar{u}^B) \cdot \frac{\partial \tau(u^C_k|\bar{u}^B)}{\partial u^B_j} \]
Determinant formula

\[ S_N = \Delta_N(\bar{u}^C) \Delta'_N(\bar{u}^B) \det_N \left[ cg^{-1}(u^C_k, \bar{u}^B) \frac{\partial \tau(u^C_k|\bar{u}^B)}{\partial u^B_j} \right] \]

Izergin–Korepin formula

\[ S_N = \sum r(\bar{u}^B_\Pi) r(\bar{u}^C_I) K_n(\bar{u}^B_\Pi|\bar{u}^C_I) K_{N-n}(\bar{u}^C_\Pi|\bar{u}^B_\Pi) f(\bar{u}^B_\Pi, \bar{u}^B_\Pi) f(\bar{u}^C_\Pi, \bar{u}^C_I) \]
Determinant formula

\[ S_N = \Delta_N(\bar{u}^C) \Delta'_N(\bar{u}^B) \det_n \left[ c g^{-1}(u^C_k, \bar{u}^B) \frac{\partial \tau(u^C_k|\bar{u}^B)}{\partial u^B_j} \right] \]

Izergin–Korepin formula

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\[ r(u^C_k) = 0 \]

\[ r(u^C_k) \rightarrow \infty \]
Determinant formula

\[ S_N = \Delta_N(\bar{u}^C) \Delta'_N(\bar{u}^B) \det \left[ t(u^C_k, u^B_j) h(u^C_k, \bar{u}^B) \right] \]

Izergin–Korepin formula

\[ \bar{u}^C_I = \emptyset \implies \bar{u}^B_I = \emptyset \]

\[ S_N = \sum r(\bar{u}^B_\Pi) r(\bar{u}^C_I) K_n(\bar{u}^B_\Pi | \bar{u}^C_I) K_{N-n}(\bar{u}^C_\Pi | \bar{u}^B_\Pi) f(\bar{u}^B_\Pi, \bar{u}^B_\Pi) f(\bar{u}^C_\Pi, \bar{u}^C_\Pi) \]

\[ r(u^C_k) = 0 \]
Determinant formula

\[ S_N = \Delta_N(\tilde{u}^C) \Delta'_N(\tilde{u}^B) \det_N \left[ t(u^C_k, u^B_j) h(u^C_k, \tilde{u}^B) \right] \]

Izergin–Korepin formula

\[ \tilde{u}_I^C = \emptyset \implies \tilde{u}_I^B = \emptyset \]

\[ S_N = r(\tilde{u}^B) K_N(\tilde{u}^C | \tilde{u}^B) \]

\[ r(u^C_k) = 0 \]
DWPF is the particular case of the scalar product.
The existence of a determinant representation for the scalar product implies the existence of a determinant representation for DWPF.
Scalar product in SU(3)-invariant models

Reshetikhin’s representation

\[ S_{a,b} = S_{a,b}(\vec{u}^C, \vec{v}^C; \vec{u}^B, \vec{v}^B) = \langle \psi_{a,b}(\vec{u}^C, \vec{v}^C) | \psi_{a,b}(\vec{u}^B, \vec{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\vec{u}_\uparrow^B)r_1(\vec{u}_\uparrow^C)r_3(\vec{v}_\uparrow^B)r_3(\vec{v}_\uparrow^C)f(\vec{u}_\uparrow^C, \vec{u}_\uparrow^C)f(\vec{u}_\uparrow^B, \vec{u}_\uparrow^B)f(\vec{v}_\uparrow^C, \vec{v}_\uparrow^C)f(\vec{v}_\uparrow^B, \vec{v}_\uparrow^B) \]

\[ \times \frac{f(\vec{v}_\downarrow^C, \vec{u}_\downarrow^C)f(\vec{v}_\downarrow^B, \vec{u}_\downarrow^B)}{f(\vec{v}_\downarrow^C, \vec{u}_\downarrow^C)f(\vec{v}_\downarrow^B, \vec{u}_\downarrow^B)} Z_{a-k,n}(\vec{u}_\downarrow^C; \vec{u}_\downarrow^B | \vec{v}_\downarrow^C; \vec{v}_\downarrow^B) Z_{k-b-n}(\vec{u}_\uparrow^B; \vec{u}_\uparrow^C | \vec{v}_\uparrow^B; \vec{v}_\uparrow^C) \]
Scalar product in SU(3)-invariant models

Reshetikhin’s representation

\[ S_{a,b} = S_{a,b}(\bar{u}_I^C, \bar{v}_I^C; \bar{u}_II^C, \bar{v}_II^C) = \langle \psi_{a,b}(\bar{u}_I^C; \bar{v}_I^C) | \psi_{a,b}(\bar{u}_II^C; \bar{v}_II^C) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}_I^B)r_1(\bar{u}_II^B)r_3(\bar{v}_I^B)r_3(\bar{v}_II^B) f(\bar{u}_I^C, \bar{u}_II^C) f(\bar{u}_II^B, \bar{u}_I^B) f(\bar{v}_II^C, \bar{v}_I^C) f(\bar{v}_I^B, \bar{v}_II^B) \]

\[ \times \frac{f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_II^B, \bar{u}_II^B)}{f(\bar{v}_I^C, \bar{u}_II^C) f(\bar{v}_II^B, \bar{u}_I^B)} Z_{a-k,n}(\bar{u}_I^C; \bar{u}_II^B | \bar{v}_I^C; \bar{v}_II^B) Z_{k-b-n}(\bar{u}_II^B; \bar{u}_I^C | \bar{v}_II^B, \bar{v}_I^C) \]

The sum is taken over partitions:

\[ \bar{u}_I^B = \{ \bar{u}_I^B, \bar{u}_II^B \} \quad \bar{v}_I^B = \{ \bar{v}_I^B, \bar{v}_II^B \} \quad \# \bar{v}_I^B = \# \bar{v}_I^C = n = 0, 1, \ldots, b \]

\[ \bar{u}_I^C = \{ \bar{u}_I^C, \bar{u}_II^C \} \quad \bar{v}_I^C = \{ \bar{v}_I^C, \bar{v}_II^C \} \quad \# \bar{u}_I^C = \# \bar{u}_II^B = k = 0, 1, \ldots, a \]
Reshetikhin Partition Function (RPF)

\[ \mathcal{Z}_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = \]

\[ (\mathcal{R}(x, y))_{jk,\ell m} = (\mathcal{R}^t_1(y, x))_{jk,\ell m} \]
Scalar product in SU(3)-invariant models

Reshetikhin’s representation

\[ S_{a,b} = S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}_1^B) r_1(\bar{u}_II^C) r_3(\bar{v}_1^B) r_3(\bar{v}_II^C) f(\bar{u}_1^C, \bar{u}_II^C) f(\bar{u}_II^B, \bar{u}_1^B) f(\bar{v}_II^C, \bar{v}_1^C) f(\bar{v}_1^B, \bar{v}_II^B) \times \frac{f(\bar{v}_II^C, \bar{u}_1^C) f(\bar{v}_1^B, \bar{u}_II^B)}{f(\bar{v}_II^C, \bar{u}_II^C) f(\bar{v}_1^B, \bar{u}_1^B)} Z_{a-k,n}(\bar{u}_II^C; \bar{u}_II^B | \bar{v}_1^C; \bar{v}_1^B) Z_{k-b-n}(\bar{u}_1^B; \bar{u}_1^C | \bar{v}_II^B; \bar{v}_II^C) \]
Scalar product in SU(3)-invariant models

Reshetikhin’s representation

\[ S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum r_1(\bar{u}_I^B) r_1(\bar{u}_II^C) r_3(\bar{v}_I^B) r_3(\bar{v}_II^C) f(\bar{u}_I^C, \bar{u}_II^C) f(\bar{u}_II^B, \bar{u}_I^B) f(\bar{v}_II^C, \bar{v}_I^C) f(\bar{v}_I^B, \bar{v}_II^B) \]

\[ \times \frac{f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_II^C, \bar{u}_II^C)}{f(\bar{v}_I^B, \bar{u}_I^B) f(\bar{v}_II^B, \bar{u}_II^B)} Z_{a-k,n}(\bar{u}_II^C; \bar{u}_II^B | \bar{v}_I^C; \bar{v}_I^B) Z_{k-b-n}(\bar{u}_I^B; \bar{u}_I^C | \bar{v}_II^B, \bar{v}_II^C) \]

Let \( |\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle \) be on-shell Bethe vector

\[ r_1(\bar{u}_I^B) = \frac{f(\bar{u}_I^B, \bar{u}_II^B)}{f(\bar{u}_II^B, \bar{u}_I^B)} f(\bar{v}_I^B, \bar{u}_I^B), \quad r_3(\bar{v}_I^B) = \frac{f(\bar{v}_II^B, \bar{v}_I^B)}{f(\bar{v}_I^B, \bar{v}_II^B)} f(\bar{v}_I^B, \bar{u}^B) \]
Scalar product in SU(3)-invariant models

Reshetikhin’s representation

\[ S_{a,b} = S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S_{a,b} = \sum_{r_1} r_1(\bar{u}_I^B) r_1(\bar{u}_I^C) r_3(\bar{v}_I^B) r_3(\bar{v}_I^C) f(\bar{u}_I^C, \bar{u}_I^C) f(\bar{u}_I^B, \bar{u}_I^B) f(\bar{v}_I^C, \bar{v}_I^C) f(\bar{v}_I^B, \bar{v}_I^B) \]

\[ \times \frac{f(\bar{v}_I^B, \bar{v}_I^B)}{f(\bar{v}_C, \bar{u}_C)} \frac{f(\bar{v}_I^B, \bar{u}_B)}{f(\bar{v}_B, \bar{u}_B)} Z_{a-k,n}(\bar{u}_I^C; \bar{u}_I^B | \bar{v}_I^C; \bar{v}_I^B) Z_{k-b-n}(\bar{u}_I^B; \bar{u}_I^C | \bar{v}_I^B; \bar{v}_I^C) \]

Let \(|\psi_{a,b}(\bar{u}^B; \bar{v}^B)\rangle\) be on-shell Bethe vector

\[ r_1(\bar{u}_I^B) = \frac{f(\bar{u}_I^B, \bar{u}_I^B)}{f(\bar{u}_B^B, \bar{u}_B^I)} f(\bar{v}_B, \bar{u}_I^B), \quad r_3(\bar{v}_I^B) = \frac{f(\bar{v}_I^B, \bar{v}_I^B)}{f(\bar{v}_I^B, \bar{v}_I^B)} f(\bar{v}_I^B, \bar{u}^B) \]
\[ S_{a,b} = \sum r_1(\bar{u}_I^B)r_1(\bar{u}_\Pi^C)r_3(\bar{v}_I^B)r_3(\bar{v}_\Pi^C)f(\bar{u}_I^C, \bar{u}_\Pi^C)f(\bar{u}_\Pi^B, \bar{u}_I^B)f(\bar{v}_\Pi^C, \bar{v}_I^C)f(\bar{v}_I^B, \bar{v}_\Pi^B) \]
\[
\times \frac{f(\bar{v}_I^C, \bar{u}_I^C)f(\bar{v}_\Pi^B, \bar{u}_\Pi^B)}{f(\bar{v}_I^C, \bar{u}_I^C)f(\bar{v}_\Pi^B, \bar{u}_\Pi^B)} Z_{a-k,n}(\bar{u}_\Pi^C; \bar{u}_\Pi^B | \bar{v}_I^C; \bar{v}_I^B) Z_{k-b-n}(\bar{u}_I^B; \bar{u}_I^C | \bar{v}_\Pi^B; \bar{v}_\Pi^C)\]

Let \( r_1(u_k^C) \to \infty \) and \( r_3(v_k^C) = 0 \). Then

\[
\bar{u}_I^C = \bar{u}_I^B = \emptyset, \quad \bar{v}_\Pi^C = \bar{v}_\Pi^B = \emptyset
\]
Special limit

\[ S_{a,b} = \sum r_1(\bar{u}_I^B) r_1(\bar{u}_\Pi^C) r_3(\bar{v}_I^B) r_3(\bar{v}_\Pi^C) f(\bar{u}_I^C, \bar{u}_\Pi^C) f(\bar{u}_\Pi^B, \bar{u}_I^B) f(\bar{v}_\Pi^C, \bar{v}_I^C) f(\bar{v}_I^B, \bar{v}_\Pi^B) \]

\[ \times \frac{f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_\Pi^B, \bar{u}_\Pi^B)}{f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_B, \bar{u}_B)} Z_{a-k,n}(\bar{u}_\Pi^C, \bar{u}_\Pi^B | \bar{v}_I^C, \bar{v}_I^B) Z_{k-b-n}(\bar{u}_I^B, \bar{u}_I^C | \bar{v}_\Pi^B, \bar{v}_\Pi^C) \]

Let \( r_1(u_k^C) \to \infty \) and \( r_3(v_k^C) = 0 \). Then

\[ \bar{u}_I^C = \bar{u}_I^B = \emptyset, \quad \bar{v}_\Pi^C = \bar{v}_\Pi^B = \emptyset \]

\[ S_{a,b} \to \frac{r_1(\bar{u}_I^C) r_3(\bar{v}_B^B)}{f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_B, \bar{u}_B)} Z_{a,b}(\bar{u}_I^C, \bar{u}_I^B | \bar{v}_I^C, \bar{v}_B^B) \]
Special limit

\[ S_{a,b} = \sum r_1(\bar{u}_I^B)r_1(\bar{u}_I^C)r_3(\bar{v}_I^B)r_3(\bar{v}_I^C) f(\bar{u}_I^C, \bar{u}_I^C) f(\bar{u}_I^B, \bar{u}_I^B) f(\bar{v}_I^C, \bar{v}_I^C) f(\bar{v}_I^B, \bar{v}_I^B) \]

\[ \times \frac{f(\bar{v}_I^C, \bar{u}_I^C)f(\bar{v}_I^B, \bar{u}_I^B)}{f(\bar{v}_I^C, \bar{u}_I^C)f(\bar{v}_I^B, \bar{u}_I^B)} \cdot Z_{a-k,n}(\bar{u}_I^B; \bar{u}_I^B | \bar{v}_I^C; \bar{v}_I^C) Z_{k-b-n}(\bar{u}_I^B; \bar{u}_I^B | \bar{v}_I^B; \bar{v}_I^B) \]

The coefficient \( Z_{a,b}(\bar{u}^C; \bar{u}^B | \bar{v}^C; \bar{v}^B) \) is the particular case of the scalar product.
Representations for RFP

\[ Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-1)^b \sum K_b(\bar{s} - c|\bar{w}_I)K_a(\bar{w}_\Pi|\bar{t})K_b(\bar{y}|\bar{w}_I)f(\bar{w}_I, \bar{w}_\Pi) \]

Here \( \bar{w} = \{\bar{s}, \bar{x}\} \). The sum is taken with respect to partitions of the set \( \bar{w} \) into subsets \( \bar{w}_I \) and \( \bar{w}_\Pi \) with \( \#\bar{w}_I = b \) and \( \#\bar{w}_\Pi = a \).
Representations for RFP

\[ Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-1)^a f(\bar{y}, \bar{x}) f(\bar{s}, \bar{t}) \sum K_a(\bar{t} - c|\bar{\eta}_I) K_a(\bar{x}|\bar{\eta}_I) K_b(\bar{\eta}_II - c|\bar{s}) f(\bar{\eta}_I, \bar{\eta}_II) \]

Here \( \bar{\eta} = \{\bar{y} + c, \bar{t}\} \). The sum is taken with respect to partitions of the set \( \bar{\eta} \) into subsets \( \bar{\eta}_I \) and \( \bar{\eta}_II \) with \( \#\bar{\eta}_I = a \) and \( \#\bar{\eta}_II = b \).
Representations for RFP

\[ Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = \sum (-1)^n f(\bar{s}, \bar{t}_I) f(\bar{y}, \bar{x}_{II}) f(\bar{t}_I, \bar{t}_{II}) f(\bar{x}_{II}, \bar{x}_I) \]

\[ \times K_n(\bar{x}_1|\bar{t}_I) K_{a-n}(\bar{x}_{II}|\bar{t}_{II} - c) K_{b+n}(\bar{y}, \bar{t}_I - c|\bar{s}, \bar{x}_I) \]

The sum is taken with respect to all partitions of the set \( \bar{t} \)
into subsets \( \bar{t}_I, \bar{t}_{II} \) and the set \( \bar{x} \) into subsets \( \bar{x}_I, \bar{x}_{II} \) with
\[ \#\bar{t}_I = \#\bar{x}_I = n, \ n = 0, 1, \ldots, a. \]
Representations for RFP

Conjecture

There is no single determinant representation for $Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y})$. 
Conjecture
There is no single determinant representation for $Z_{a,b}(\bar{t}; \bar{x} | \bar{s}; \bar{y})$.

Corollary
There is no single determinant representation for the scalar products of on-shell Bethe vectors with arbitrary Bethe vectors.
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There is no single determinant representation for the scalar products of on-shell Bethe vectors with arbitrary Bethe vectors.

We can consider special scalar products involving on-shell Bethe vector and some specific off-shell Bethe vector. In particular the scalar product of on-shell vector and twisted on-shell vector.
Corollary

There is no single determinant representation for the scalar products of on-shell Bethe vectors with arbitrary Bethe vectors.

We can consider special scalar products involving on-shell Bethe vector and some specific off-shell Bethe vector. In particular the scalar product of on-shell vector and twisted on-shell vector.

\[
S^{(\kappa)}_{a,b} \equiv S^{(\kappa)}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi^{(\kappa)}_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle
\]

\[
r_1(u^C_k) = \kappa \frac{f(u^C_k, \bar{u}^C_k)}{f(\bar{u}^C_k, u^C_k)} f(\bar{v}^C, u^C_k), \quad r_3(v^C_k) = \kappa \frac{f(\bar{v}^C_k, v^C_k)}{f(v^C_k, \bar{v}^C_k)} f(v^C_k, \bar{u})
\]

\[
r_1(u^B_k) = \frac{f(u^B_k, \bar{u}^B_k)}{f(\bar{u}^B_k, u^B_k)} f(\bar{v}^B, u^B_k), \quad r_3(v^B_k) = \frac{f(\bar{v}^B_k, v^B_k)}{f(v^B_k, \bar{v}^B_k)} f(v^B_k, \bar{u}^B)
\]
The SU(2) analog of this scalar product is the generating function of $\sigma^z$-form factors in XXX SU(2)-invariant Heisenberg chain

$$Q_\kappa(m) = \frac{\tau^m(0|\bar{u}^C)}{\tau^m(0|\bar{u}^B)} \langle \psi_n^{(\kappa)}(\bar{u}^C)|\psi_n(\bar{u}^B) \rangle$$

$$\langle \psi_n(\bar{u}^C)|(1 - \sigma^z_m)|\psi_n(\bar{u}^B) \rangle = 2 \frac{d}{d\kappa}(Q_\kappa(m) - Q_\kappa(m - 1))\bigg|_{\kappa=1}$$
The SU(2) analog of this scalar product is the generating function of $\sigma^z$-form factors in XXX SU(2)-invariant Heisenberg chain

$$Q_\kappa(m) = \frac{\tau^m_\kappa(0|\bar{u}^C)}{\tau^m(0|\bar{u}^B)} \langle \psi_{\bar{u}}^{(\kappa)}(\bar{u}^C)|\psi_{\bar{u}}^n(\bar{u}^B) \rangle$$

$$\langle \psi_{\bar{u}}(\bar{u}^C)| (1 - \sigma^z_m) |\psi_{\bar{u}}(\bar{u}^B) \rangle = 2 \frac{d}{d\kappa}(Q_\kappa(m) - Q_\kappa(m - 1)) \bigg|_{\kappa=1}$$

Similar formula exists for XXX SU(3)-invariant Heisenberg chain

$$Q_\kappa(m) = \frac{\tau^m_\kappa(0|\bar{u}^C; \bar{v}^C)}{\tau^m(0|\bar{u}^B; \bar{v}^B)} \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C)|\psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle$$

$$\langle \psi_{\bar{u}}(\bar{u}^C)|E_{m}^{2,2}|\psi_{\bar{u}}(\bar{u}^B) \rangle = \frac{d}{d\kappa}(Q_\kappa(m) - Q_\kappa(m - 1)) \bigg|_{\kappa=1}$$

where elementary units $E_m^{\epsilon,\epsilon'}$ are: $$(E_m^{\epsilon,\epsilon'})_{jk} = \delta_{j\epsilon}\delta_{k\epsilon'}$$
Determinant representation for SU(2) scalar product

\[ S_n^{(\kappa)}(\bar{u}^C; \bar{u}^B) = \langle \psi_n^{(\kappa)}(\bar{u}^C)|\psi_n(\bar{u}^B) \rangle = \Delta'_n(\bar{u}^C)\Delta_n(\bar{u}^B) \det_n \mathcal{M} \]

\[ = \Delta'_n(\bar{u}^C)\Delta_n(\bar{u}^B) \det_n \mathcal{M}_\kappa \]

\[ \mathcal{M} \text{ and } \mathcal{M}_\kappa \text{ are matrices of the size } n \times n \]

\[ \mathcal{M}_\kappa = cg^{-1}(u_k^C, \bar{u}^B) \frac{\partial \tau_\kappa(u_k^B)}{\partial u_j^C} \]

\[ \mathcal{M} = cg^{-1}(u_j^B, \bar{u}^C) \frac{\partial \tau(u_j^C)}{\partial u_k^B} \]
Determinant representation for SU(2) scalar product

\[
S_n^{(\kappa)}(\bar{u}^C; \bar{u}^B) = \langle \psi_n^{(\kappa)}(\bar{u}^C) | \psi_n(\bar{u}^B) \rangle = \Delta'_n(\bar{u}^C)\Delta_n(\bar{u}^B) \det_n \mathcal{M} = \Delta'_n(\bar{u}^C)\Delta_n(\bar{u}^B) \det_n \mathcal{M}_\kappa
\]

\(\mathcal{M}\) and \(\mathcal{M}_\kappa\) are matrices of the size \(n \times n\)

\[
\mathcal{M}_\kappa = cg^{-1}(u^C_k, \bar{u}^B) \frac{\partial \tau_k(u^B_k)}{\partial u^C_j} \quad \mathcal{M} = cg^{-1}(u^B_j, \bar{u}^C) \frac{\partial \tau(u^C_j)}{\partial u^B_k}
\]
Determinant representation for SU(2) scalar product

\[ S^{(\kappa)}_n(\bar{u}^C; \bar{u}^B) = \langle \psi^{(\kappa)}_n(\bar{u}^C) | \psi_n(\bar{u}^B) \rangle = \Delta'_n(\bar{u}^C) \Delta_n(\bar{u}^B) \det_n M = \Delta'_n(\bar{u}^C) \Delta_n(\bar{u}^B) \det_n M_\kappa \]

\( M \) and \( M_\kappa \) are matrices of the size \( n \times n \)

\[ M_\kappa = cg^{-1}(u^C_k, \bar{u}^B) \frac{\partial \tau_\kappa(u^B_k)}{\partial u^C_j} \quad M = cg^{-1}(u^B_j, \bar{u}^C) \frac{\partial \tau(u^C_j)}{\partial u^B_k} \]

\[ \tau_\kappa(w|\bar{u}^C) = r(w)f(\bar{u}^C, w) + \kappa f(w, \bar{u}^C) \]

\[ \tau(w|\bar{u}^B) = r(w)f(\bar{u}^B, w) + f(w, \bar{u}^B) \]
Scalar product of on-shell vector and twisted on-shell vector (SU(3))

\[ S_{a,b}^{(\kappa)} \equiv S_{a,b}^{(\kappa)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]
Scalar product of on-shell vector
and twisted on-shell vector \((\text{SU}(3))\)

\[ S_{a,b}^{(\kappa)} \equiv S_{a,b}^{(\kappa)}(\vec{u}^C, \vec{v}^C; \vec{u}^B, \vec{v}^B) = \langle \psi_{a,b}^{(\kappa)}(\vec{u}^C; \vec{v}^C) | \psi_{a,b}(\vec{u}^B; \vec{v}^B) \rangle \]

\[ S_{a,b}^{(\kappa)} = \sum r_1(\vec{u}^B_1)r_1(\vec{u}^C_\Pi)r_3(\vec{v}^B_3)f(\vec{u}^C_1, \vec{u}^C_\Pi)f(\vec{u}^B_3, \vec{u}^B_1)f(\vec{v}^C_3, \vec{v}^C_\Pi)f(\vec{v}^B_3, \vec{v}^B_1) \]

\[ \times \frac{f(\vec{u}^C_1, \vec{u}^C_\Pi)f(\vec{v}^B_3, \vec{u}^B_1)}{f(\vec{v}^C, \vec{u}^C)f(\vec{v}^B, \vec{u}^B)} Z_{a-k,n}(\vec{u}^C_\Pi; \vec{u}^B_1 | \vec{v}^C_1; \vec{v}^B_1) Z_{k,b-n}(\vec{u}^B_1; \vec{u}^C_\Pi | \vec{v}^B_3; \vec{v}^B_3) \]
Scalar product of on-shell vector
and twisted on-shell vector (SU(3))

\[ S^{(\kappa)}_{a,b} \equiv S^{(\kappa)}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi^{(\kappa)}_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S^{(\kappa)}_{a,b} = \sum r_1(\bar{u}^B) r_1(\bar{u}^C) r_3(\bar{v}^B) r_3(\bar{v}^C) f(\bar{u}^C, \bar{u}^C) f(\bar{u}^B, \bar{u}^B) f(\bar{v}^C, \bar{v}^C) f(\bar{v}^B, \bar{v}^B) \]

\[ \times \frac{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} Z_{a-k,n}(\bar{u}^C_\Pi; \bar{u}^B_\Pi | \bar{v}^C_\Pi; \bar{v}^B_\Pi) Z_{k-b-n}(\bar{u}^B_\Pi; \bar{u}^C_\Pi | \bar{v}^B_\Pi; \bar{v}^C_\Pi) \]

\[ r_1(u^C_k) = \kappa \frac{f(u^C_k, \bar{u}^C_k)}{f(\bar{u}^C_k, u^C_k)} f(\bar{v}^C, u^C_k), \quad r_3(v^C_k) = \kappa \frac{f(\bar{v}^C_k, v^C_k)}{f(v^C_k, \bar{v}^C_k)} f(v^C_k, \bar{u}) \]

\[ r_1(u^B_k) = \frac{f(u^B_k, \bar{u}^B_k)}{f(\bar{u}^B_k, u^B_k)} f(\bar{v}^B, u^B_k), \quad r_3(v^B_k) = \frac{f(\bar{v}^B_k, v^B_k)}{f(v^B_k, \bar{v}^B_k)} f(v^B_k, \bar{u}) \]
Scalar product of on-shell vector and twisted on-shell vector (SU(3))

\[ S^{(\kappa)}_{a,b} \equiv S^{(\kappa)}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \langle \psi^{(\kappa)}_{a,b}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

\[ S^{(\kappa)}_{a,b} = \sum_{r_1} r_1(\bar{u}^B_1) r_1(\bar{u}^C_\Pi) r_3(\bar{v}^B_\Pi) r_3(\bar{v}^C_\Pi) f(\bar{u}^C_1, \bar{u}^C_\Pi) f(\bar{u}^B_\Pi, \bar{u}^B_1) f(\bar{v}^C_\Pi, \bar{v}^C_1) f(\bar{v}^B_1, \bar{v}^B_\Pi) \]

\[ \times \frac{f(\bar{v}^C_1, \bar{u}^C_1) f(\bar{v}^B_\Pi, \bar{u}^B_\Pi)}{f(\bar{v}^C_\Pi, \bar{u}^C_\Pi) f(\bar{v}^B_1, \bar{u}^B_1)} Z_{a-k,n}(\bar{u}^B_\Pi; \bar{v}^C_\Pi; \bar{v}^B_1) Z_{k-b-n}(\bar{u}^B_1; \bar{u}^C_1; \bar{v}^B_\Pi; \bar{v}^C_\Pi) \]

\[ r_1(u^C_k) = \kappa \frac{f(u^C_k, \bar{u}^C_k)}{f(\bar{u}^C_k, u^C_k)} f(\bar{v}^C, u^C_k), \quad r_3(v^C_k) = \kappa \frac{f(\bar{v}^C_k, v^C_k)}{f(v^C_k, \bar{v}^C_k)} f(v^C_k, \bar{u}) \]

\[ r_1(u^B_k) = \frac{f(u^B_k, \bar{u}^B_k)}{f(\bar{u}^B_k, u^B_k)} f(\bar{v}^B, u^B_k), \quad r_3(v^B_k) = \frac{f(\bar{v}^B_k, v^B_k)}{f(v^B_k, \bar{v}^B_k)} f(v^B_k, \bar{u}) \]
Determinant representation for SU(3) scalar product

\[ S^{(k)}_{a,b} = t(\bar{v}^C, \bar{u}^B) \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \det \mathcal{N} \]

\( \mathcal{N} \) is a block-matrix of the size \((a + b) \times (a + b)\)

\[ \mathcal{N} = \begin{pmatrix}
\mathcal{N}(u) (u^C_j, u^B_k) & \mathcal{N}(u) (u^C_j, v^C_k) \\
\mathcal{N}(v) (v^B_j, u^B_k) & \mathcal{N}(v) (v^B_j, v^C_k)
\end{pmatrix} \]
Determinant representation for SU(3) scalar product

\[
S^{(\kappa)}_{a,b} = t(\bar{v}^C, \bar{u}^B) \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \det \mathcal{N}_{a+b}
\]

\( \mathcal{N} \) is a block-matrix of the size \((a + b) \times (a + b)\)

\[
\mathcal{N} = \begin{pmatrix}
\mathcal{N}^{(u)}(u_j^C, u_k^B) & \mathcal{N}^{(u)}(u_j^C, v_k^C) \\
\mathcal{N}^{(v)}(v_j^B, u_k^B) & \mathcal{N}^{(v)}(v_j^B, v_k^C)
\end{pmatrix}
\]

\[
\mathcal{N}^{(u)}(u_j^C, w_k) = c g^{-1}(w_k, \bar{u}^C) g^{-1}(\bar{v}^C, w_k) \frac{\partial \tau_k(w_k)}{\partial u_j^C}, \quad w_k = u_k^B, v_k^C
\]

\[
\mathcal{N}^{(v)}(v_j^B, w_k) = -c g^{-1}(\bar{v}^B, w_k) g^{-1}(w_k, \bar{u}^B) \frac{\partial \tau(w_k)}{\partial v_j^B}, \quad w_k = u_k^B, v_k^C
\]
Determinant representation for SU(3) scalar product

\[ S_{a,b}^{(\kappa)} = t(\bar{v}^C, \bar{u}^B) \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \det \mathcal{N}_{a+b} \]

\( \mathcal{N} \) is a block-matrix of the size \((a + b) \times (a + b)\)

\[
\mathcal{N} = 
\begin{pmatrix}
\mathcal{N}^{(u)}(u_j^C, u_k^B) & \mathcal{N}^{(u)}(u_j^C, v_k^C) \\
\mathcal{N}^{(v)}(v_j^B, u_k^B) & \mathcal{N}^{(v)}(v_j^B, v_k^C)
\end{pmatrix}
\]

\[
\mathcal{N}^{(u)}(u_j^C, w_k) = cg^{-1}(w_k, \bar{u}^C)g^{-1}(\bar{v}^C, w_k) \frac{\partial \tau^{(\kappa)}(w_k)}{\partial u_j^C}, \quad w_k = u_k^B, v_k^C
\]

\[
\mathcal{N}^{(v)}(v_j^B, w_k) = -cg^{-1}(\bar{v}^B, w_k)g^{-1}(w_k, \bar{u}^B) \frac{\partial \tau(w_k)}{\partial v_j^B}, \quad w_k = u_k^B, v_k^C
\]
\[ \langle \psi^{(\kappa)}_{a,b} (\bar{u}^C; \bar{v}^C) | \psi_{a,b} (\bar{u}^B; \bar{v}^B) \rangle \rightarrow \langle \psi_{a,b} (\bar{u}^C; \bar{v}^C) | E_{m_i}^{2,2} | \psi_{a,b} (\bar{u}^B; \bar{v}^B) \rangle \]
Is it possible to calculate other form factors?

\[ F_{\epsilon,\epsilon'}^{m} = \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C) | E_{m}^{\epsilon,\epsilon'} | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle = ? \]
Perspectives

\[ \langle \psi_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \rightarrow \langle \psi_{a,b}(\bar{u}^C; \bar{v}^C) | E_m^2 | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]

Is it possible to calculate other form factors?

\[ F_{\epsilon,\epsilon'}^{m} = \langle \psi_{a',b'}^{(\kappa)}(\bar{u}^C; \bar{v}^C) | E_m^\epsilon,\epsilon' | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle = ? \]

Inverse scattering problem

\[ E_m^\epsilon,\epsilon' = T^{m-1}(0) T_{\epsilon',\epsilon}(0) T^{-m}(0) \]

\[ F_{\epsilon,\epsilon'}^{m} = \frac{\tau^{m-1}(0|\bar{u}^C, \bar{v}^C)}{\tau_m(0|\bar{u}^B, \bar{v}^B)} \langle \psi_{a',b'}(\bar{u}^C; \bar{v}^C) | T_{\epsilon',\epsilon}(0) | \psi_{a,b}(\bar{u}^B; \bar{v}^B) \rangle \]
Perspectives

\[ F^{2,2}_m \sim \frac{d}{d\kappa} \det_{a+b} \begin{pmatrix} (*) \frac{\partial \tau_\kappa}{\partial u^C_j} \\ \vdots \\ (*) \frac{\partial \tau}{\partial v^B_j} \end{pmatrix} \bigg|_{\kappa=1} \]
After setting $\kappa = 1$ the eigenvalue $\tau_\kappa$ turns into $\tau$
After setting $\kappa = 1$ the eigenvalue $\tau_\kappa$ turns into $\tau$

$F_{m}^{2,2} \sim \frac{d}{d\kappa} \det_{a+b} \left( \begin{array}{cccc} \left(\ast\right) \frac{\partial \tau_\kappa}{\partial u_j^C} \\ \left(\ast\right) \frac{\partial \tau_{\kappa}}{\partial v_j^B} \end{array} \right)_{\kappa=1}$

$F_{m}^{\epsilon,\epsilon'} \sim \det_{a+b} \left( \begin{array}{c} \left(\ast\right) \frac{\partial \tau}{\partial s_j} \\ \end{array} \right)$

$s_j = u_j^C, v_j^B$

modified row
Perspectives

\[ F_{m,2}^{2,2} \sim \frac{d}{d\kappa} \det_{a+b} \begin{pmatrix} \partial_{\tau_{\kappa}} \frac{\partial C_j}{\partial u_j} \\ \partial_{\tau} \frac{\partial B_j}{\partial v_j} \end{pmatrix} \bigg|_{\kappa=1} \]

After setting \( \kappa = 1 \) the eigenvalue \( \tau_{\kappa} \) turns into \( \tau \)

\[ F_{m}^{\epsilon,\epsilon'} \sim \det_{a+b} \begin{pmatrix} \partial_{\tau} \frac{\partial}{\partial s_j} \end{pmatrix} \]

\( s_j = u_j^{\epsilon}, v_j^{\epsilon'} \)
Perspectives

We should calculate the action of $T_{ij}(w)$ on Bethe vectors

$$T_{ij}(w)|\psi_{a,b}(\bar{u}; \bar{v})\rangle = |\phi(w|\bar{u}; \bar{v})\rangle$$

Is it possible to present the result of this action as a linear combination of off-shell Bethe vectors?

$$|\phi(w|\bar{u}; \bar{v})\rangle = \sum_{k} \alpha_{k} |\psi_{a',b'}(\bar{u}^{(k)}; \bar{v}^{(k)})\rangle$$
Perspectives

We should calculate the action of $T_{ij}(w)$ on Bethe vectors

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$$|\phi(w|\bar{u}; \bar{v})\rangle = \sum_k \alpha_k |\psi_{a',b'}(\bar{u}^{(k)}; \bar{v}^{(k)})\rangle$$

$$|\psi_{a,b}(\bar{u}; \bar{v})\rangle = P(T_{ij}(u_k), T_{ij}(v_k))|0\rangle$$

It is not apriori obvious that $|\psi_{a,b}(\bar{u}; \bar{v})\rangle$ can be presented as a linear combination of the polynomials of the same type.
Summation over partitions

- We use different representations for RPF
  \[ Z_{a-k,n}(\bar{u}^C; \bar{u}^B|\bar{v}^C; \bar{v}^B) \text{ and } Z_{k,b-n}(\bar{u}^B; \bar{u}^C|\bar{v}^B; \bar{v}^C) \].

- We use summation identities for DWPF \( K_n(\bar{x}|\bar{y}) \)

\[
\sum K_{m_1}(\bar{z}_I|\bar{x})K_{m_2}(\bar{y}|\bar{z}_II)f(\bar{z}_II, \bar{z}_I) = (-1)^{m_1}f(\bar{z}, \bar{x})K_{m_1+m_2}(\bar{x} - c, \bar{y}|\bar{z})
\]

The sum is taken over partitions of the set \( \bar{z} \) into subsets \( \bar{z}_I \) and \( \bar{z}_II \) with \( \#\bar{z}_I = m_1 \) and \( \#\bar{z}_II = m_2 \).
Summation over partitions

- We use different representations for RPF
  \[ Z_{a-k,n}(\bar{u}_I^C; \bar{u}_I^B|\bar{v}_I^C; \bar{v}_I^B) \text{ and } Z_{k,b-n}(\bar{u}_I^B; \bar{u}_I^C|\bar{v}_I^B; \bar{v}_I^C). \]

- We use summation identities for DWPF \( K_n(\bar{x}|\bar{y}) \)

\[
\sum f(\bar{y}_I, \bar{y}_I) f(\bar{x}_I, \bar{x}_I) K_{m_I}(\bar{y}_I|\bar{x}_I) K_{m_I}(\bar{x}_II + c|\bar{y}_I) = (-1)^t h(\bar{x}, \bar{x}) h(\bar{y}, \bar{y})
\]

The sum is taken over all possible partitions of the sets \( \bar{x} \) and \( \bar{y} \)
with \( \#\bar{x}_I = \#\bar{y}_I = m_I \) and \( \#\bar{x}_II = \#\bar{y}_II = m_{II} \).
Conjecture

\[ S_{a,b} = \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \det N_{a+b} \]

\( N \) is a block-matrix of the size \((a + b) \times (a + b)\)

\[
N = \begin{pmatrix}
    (\ast) \frac{\partial \tau(u_k^C)}{\partial u_j^B} & (\ast) \frac{\partial \tau(v_k^C)}{\partial u_j^B} \\
    (\ast) \frac{\partial \tau(u_k^C)}{\partial v_j^B} & (\ast) \frac{\partial \tau(v_k^C)}{\partial v_j^B}
\end{pmatrix}
\]
Conjecture

\[ S_{a,b} = \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \det \mathcal{N} \]

\( \mathcal{N} \) is a block-matrix of the size \( (a + b) \times (a + b) \)

\[
\mathcal{N} = \begin{pmatrix}
\left( \ast \right) \frac{\partial \tau(u^C_k)}{\partial u^B_j} & \left( \ast \right) \frac{\partial \tau(v^C_k)}{\partial u^B_j} \\
\left( \ast \right) \frac{\partial \tau(u^C_k)}{\partial v^B_j} & \left( \ast \right) \frac{\partial \tau(v^C_k)}{\partial v^B_j}
\end{pmatrix}
\]

Simple tests for small \( a \) and \( b \) do not confirm this conjecture.