Multiple Schramm–Loewner evolutions for conformal field theories with Lie algebra symmetries

(arXiv:1207.4057v3)

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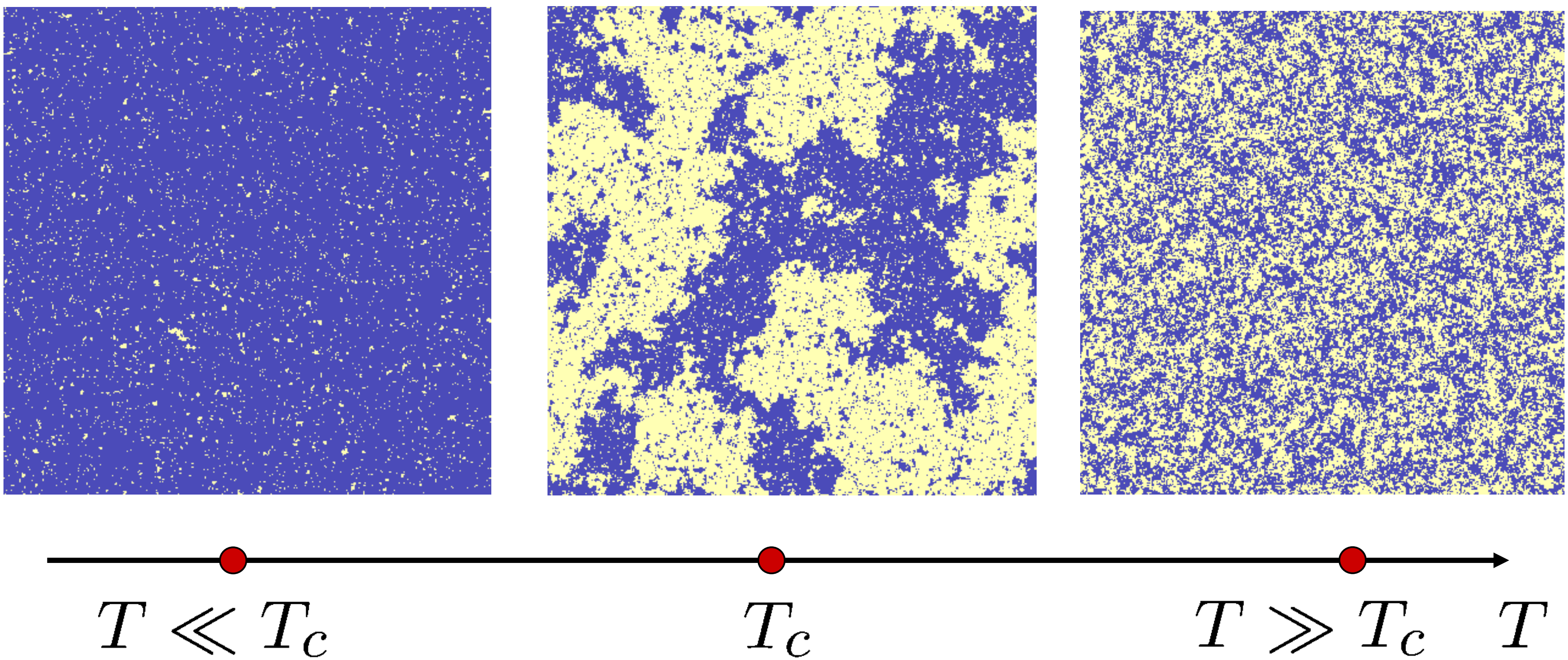
Geometric Aspects of Critical Phenomena

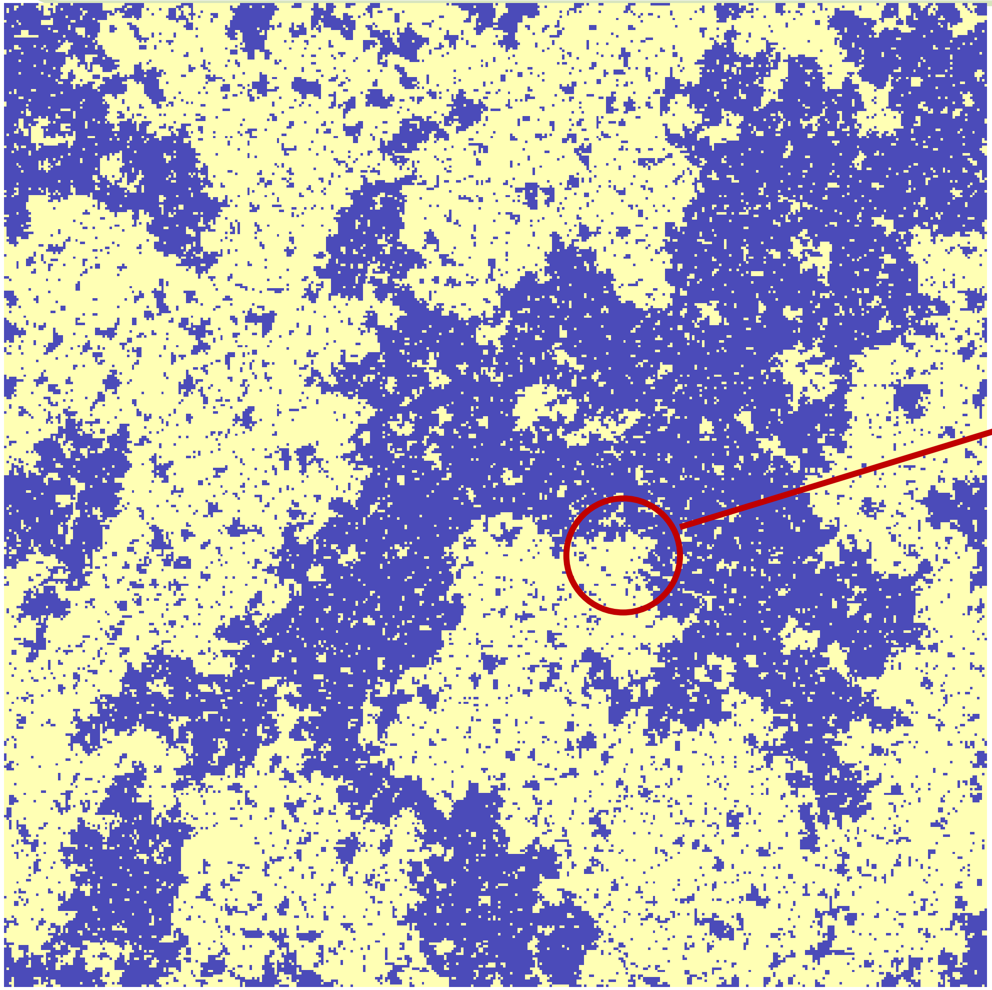
e.g. 2D Ising model $\mathcal{H} = -J \sum_{\langle jk \rangle} \sigma_j \sigma_k \quad (J > 0)$



$$T_c = \frac{2J}{\log(1 + \sqrt{2})}$$

$T = T_c$

Critical Phenomena (Conformal Invariance)



2D Ising modl $T = T_c$

Spin Cluster Boundaries

Random Curves

Fractal Dim. $d_f = \frac{11}{8}$

(Saleur and Duplantier ('87))

- Potts model,
- critical percolation, ・・・

Random curves of various models in 2D are directly described by SLE.

SLE (Schramm-Loewner Evolution)

Schramm ('00)

$$dg_t(z) = \frac{2dt}{g_t(z) - x_t}, \quad g_0(z) = z \in \mathbb{H}$$

$$g_t(z) = z + O(1) \quad (z \to \infty)$$

$$x_t = \sqrt{\kappa}\xi_t \qquad \xi_t$$: **Standard Brownian Motion**

$$\mathbf{E}[dx_t] = 0 \text{ and } \mathbf{E}[dx_t dx_t] = \kappa dt$$

- SLE has a solution up to τ_z . τ_z : first time when $g_t(z)$ hits x_t .
- $g_t^{-1}(x_t)$ defines the tip γ_t of the random curve.
- Hull $K_t = \overline{\{z \in \mathbb{H} | \tau_z < t\}}$. K_t is a increasing family of hulls: $K_s \subset K_t$ for $s < t$.

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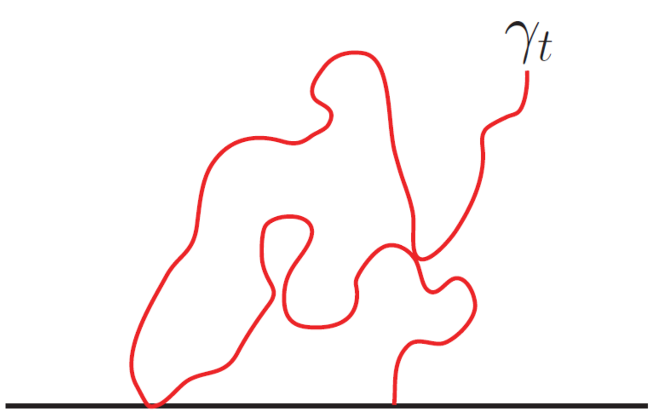
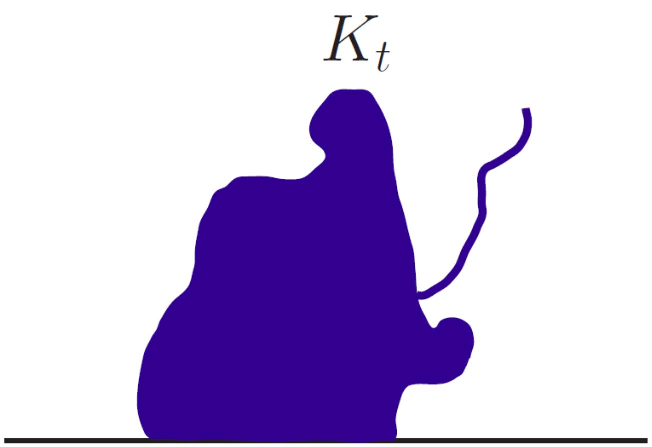


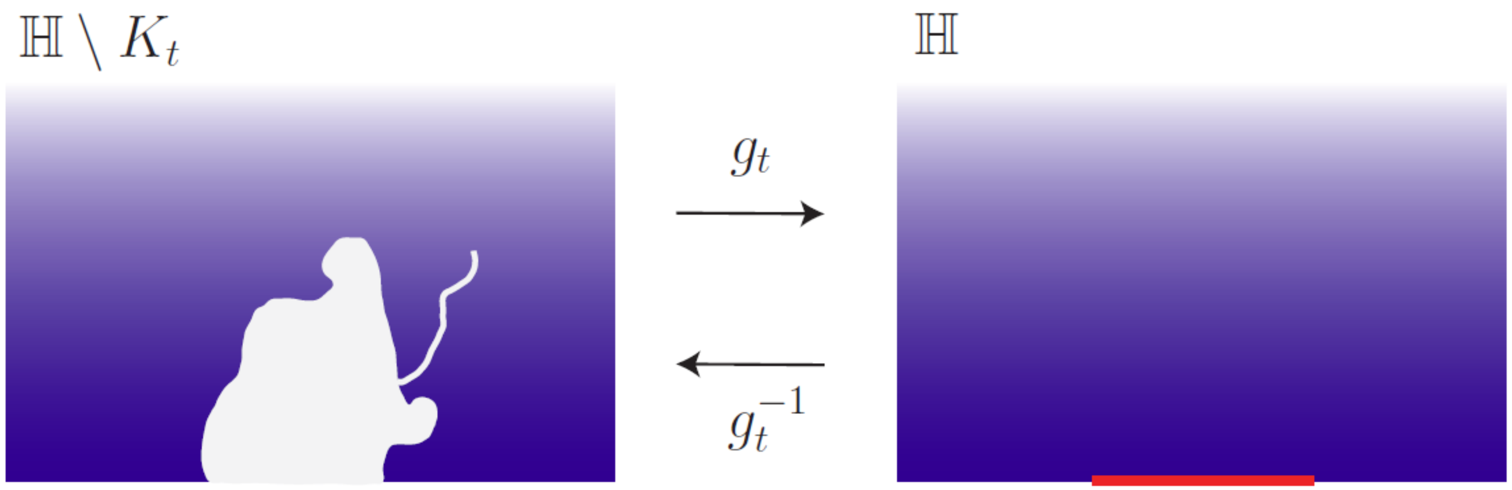


$$dg_t(z) = \frac{2dt}{g_t(z) - x_t}, \quad g_0(z) = z \in \mathbb{H}$$

$$g_t(z) = z + O(1) \quad (z \to \infty)$$

$$x_t = \sqrt{\kappa}\xi_t$$ ξ_t : **Standard Brownian Motion**

- $g_t(z)$ is the unique conformal map uniformizing the complement of the hull K_t.



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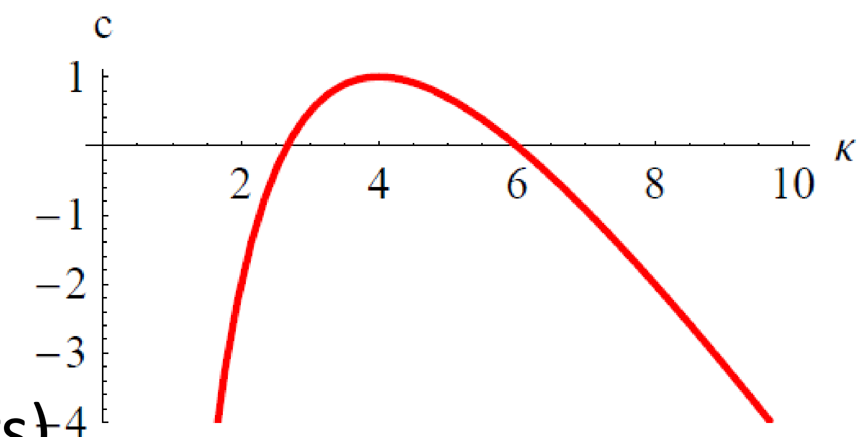
$$x_t = \sqrt{\kappa}\xi_t \qquad \xi_t :$$ **Standard Brownian Motion**

- The connection between SLE and CFT (conformal field theory) is well understood.

Bauer and Bernard ('02)
Friedrich and Werner ('02)

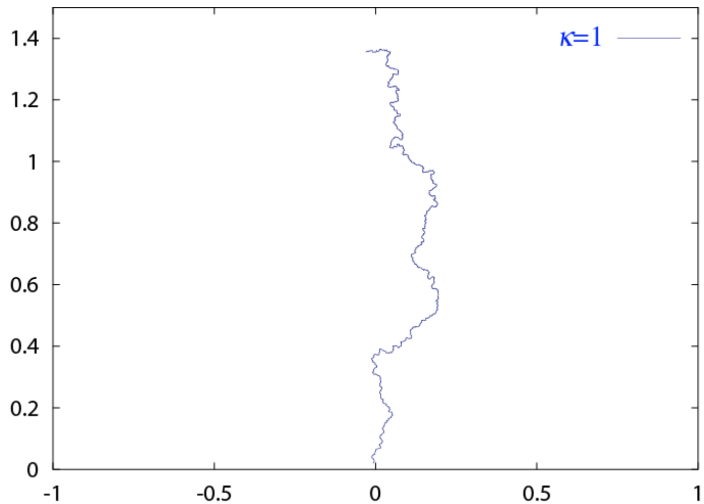
$$c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}$$



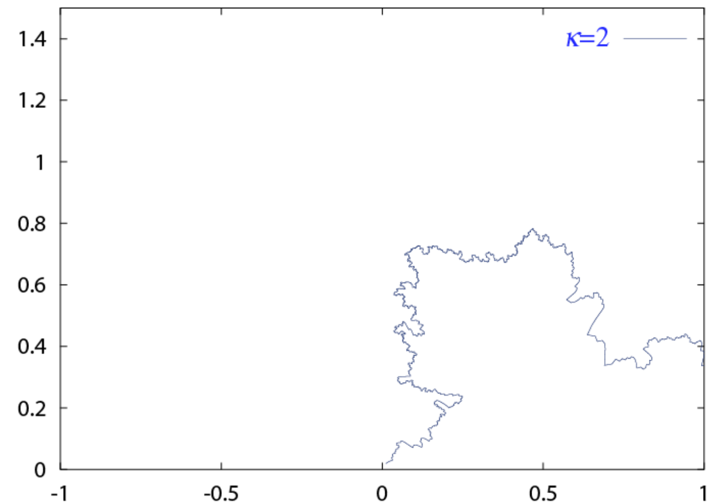
- κ=6 (c=0): critical percolation
- κ=3 (c=1/2): Ising (spin clusters)
- κ=16/3 (c=1/2): Ising (FK clusters)

$$x_t = \sqrt{\kappa}\xi_t$$

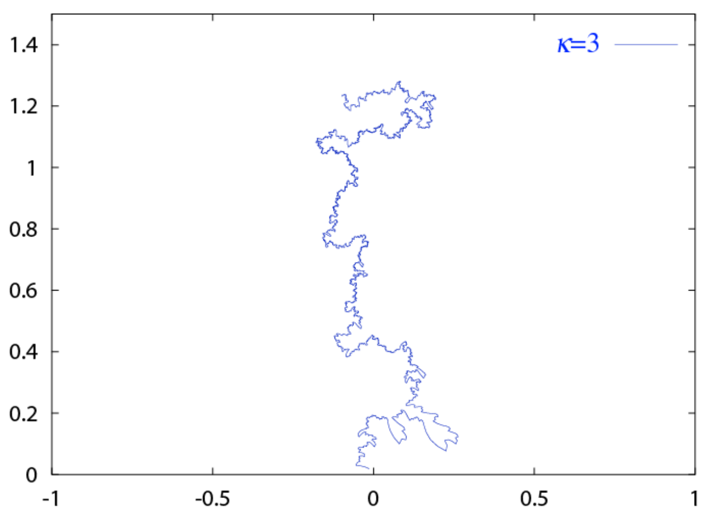
$$\boxed{d_f = 1 + \kappa/8}$$



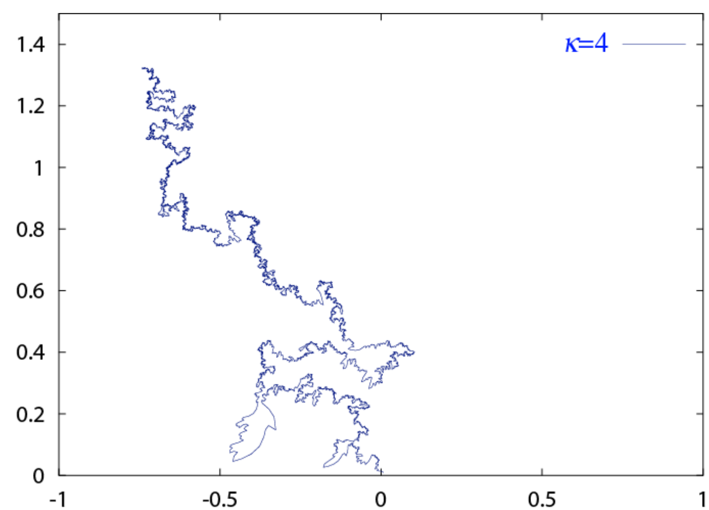
$\kappa = 1$ $(d_f = 9/8)$



$\kappa = 2$ $(d_f = 5/4)$



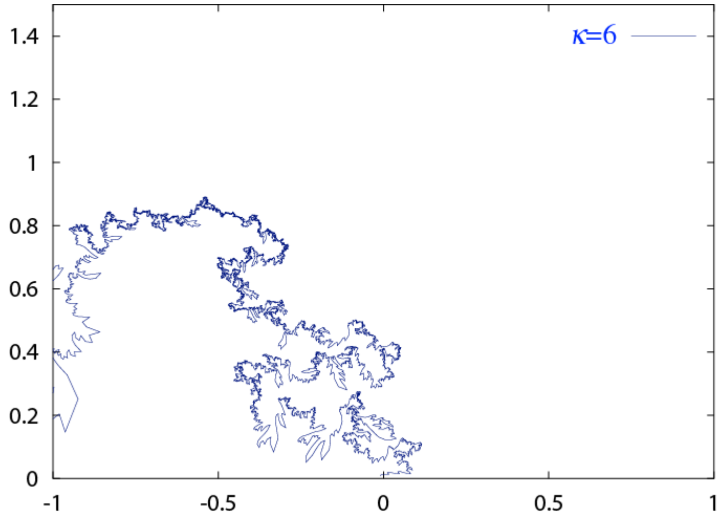
$\kappa = 3$ $(d_f = 11/8)$ **Ising**



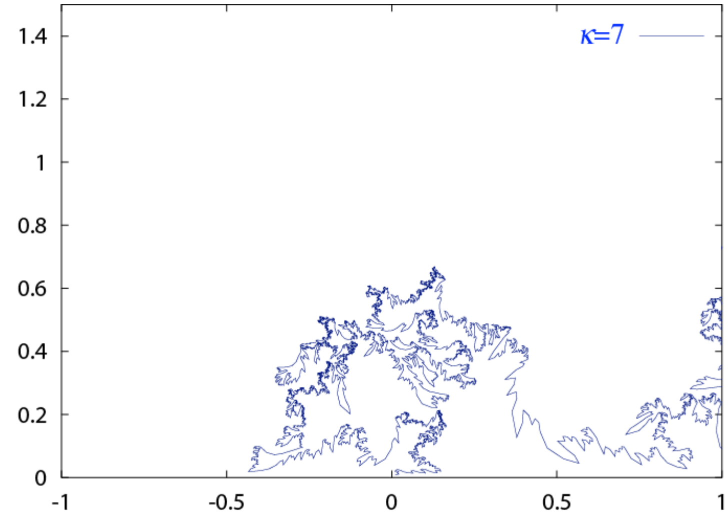
$\kappa = 4$ $(d_f = 3/2)$

$$x_t = \sqrt{\kappa}\xi_t$$

$$\boxed{d_f = 1 + \kappa/8}$$

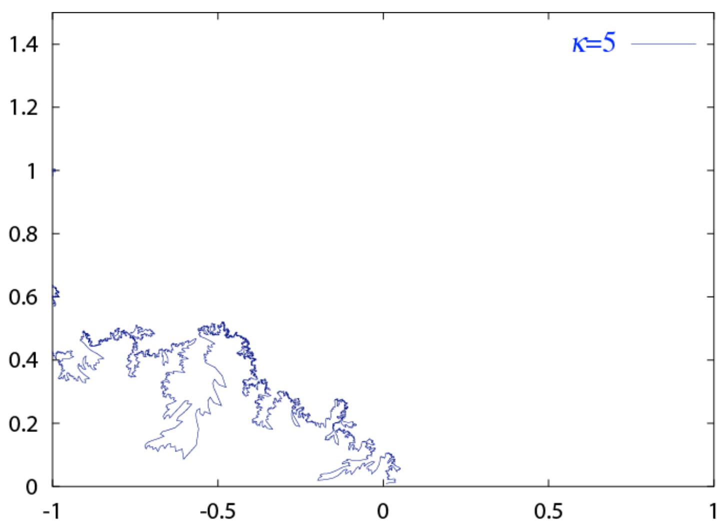


$\kappa = 5 \ (d_f = 13/8)$

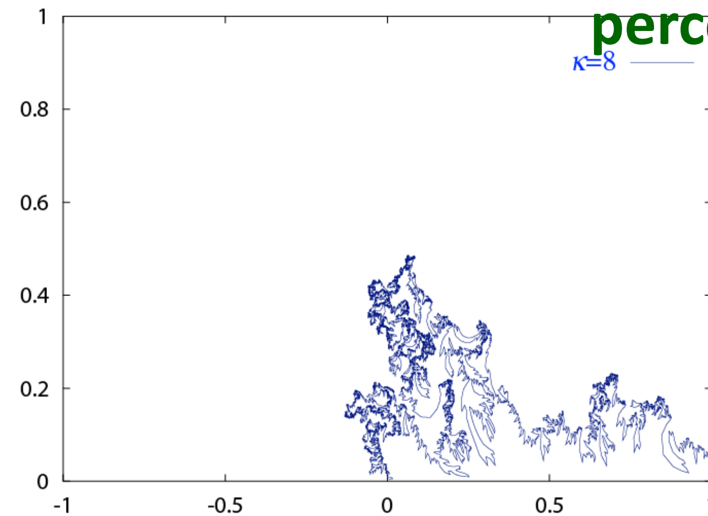


$\kappa = 6 \ (d_f = 7/4)$

critical percolation

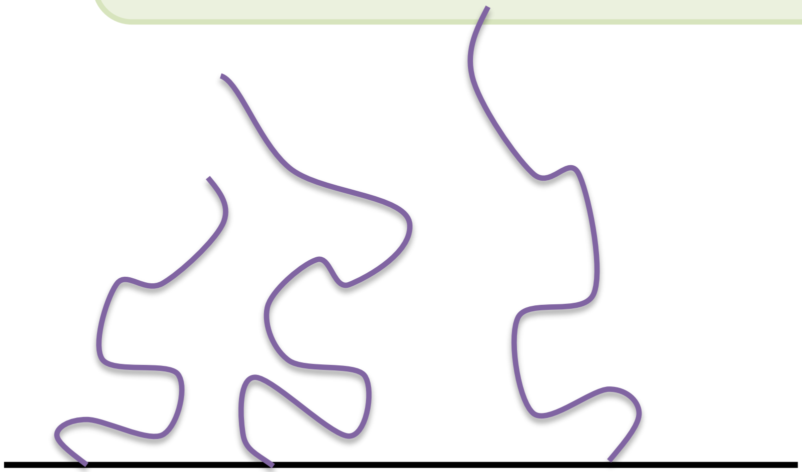


$\kappa = 7 \ (d_f = 15/8)$



$\kappa = 8 \ (d_f = 2)$

Multiple + Extra Symmetries



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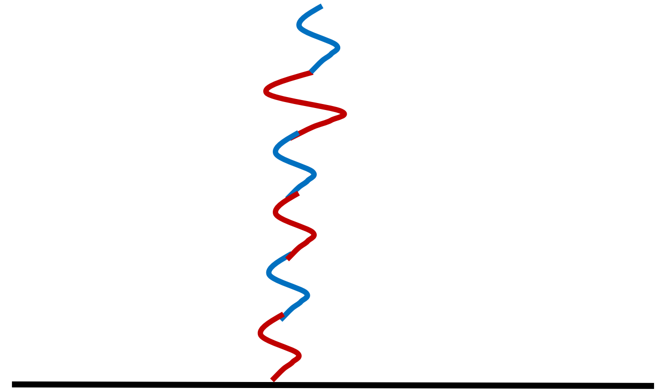

multiple SLE

multiple random curves

Cardy (02)
Dubedat (04)
Bauer, Bernard, Kytola (05)

SLE with "spin"

geometric random walks +
"algebraic" random walks

Bettelheim et al (05)
Santachiara (08)
Alekseev, Bytsko, Izyurov (11)

Construct the SLE for the WZW models

Derive evolutions so that they are compatible with CFT.

Geometric part

$$dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}$$

dq_α : time intervals $\quad \sum_\alpha dq_\alpha = dt$

$$dx_{\alpha t} = \sqrt{\kappa}d\xi_{\alpha t} + \underline{dF_{\alpha t}}$$ drift term

$\xi_{\alpha t}$: $\mathbb{R}^m$-valued Brownian motion

$$\mathbf{E}[d\xi_{\alpha t}] = 0, \quad \mathbf{E}[d\xi_{\alpha t} d\xi_{\beta t}] = \delta_{\alpha\beta} dq_\alpha$$

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Geometric part

$$dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}$$

algebraic part

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Stochastic process living on a Lie group manifold

$$\exp[\textstyle\sum_a d\theta_t^a(z) t^a]$$

t^a : generators of $\mathfrak{g}$

$a = 1, \ldots, \dim \mathfrak{g}$

$$dp_{\alpha t}^a = \sqrt{\tau} d\vartheta_{\alpha t}^a + \underline{dG_{\alpha t}^a}$$

drift term

$\vartheta_{\alpha t}^a$: $\mathbb{R}^{m \dim \mathfrak{g}}$-valued Brownian motion

$$\mathbf{E}[d\vartheta_{\alpha t}^a] = 0, \quad \mathbf{E}[d\vartheta_{\alpha t}^a d\vartheta_{\beta t}^b] = \delta^{ab} \delta_{\alpha\beta} dq_\alpha$$

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drift term

Drift terms are determined by "SLE martingale".

$\vartheta_{\alpha t}^a$: $\mathbb{R}^{m \dim \mathfrak{g}}$-valued Brownian motion

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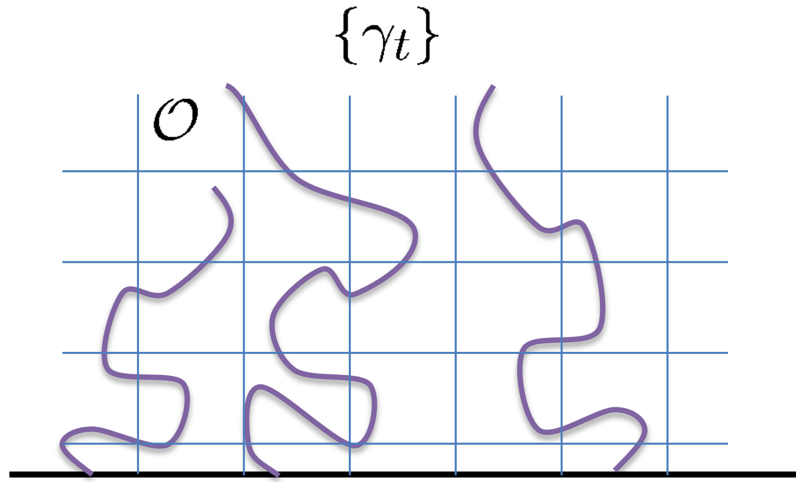
SLE/CFT correspondence

(Bauer and Bernard)

Consider the corresponding statistical mechanics model on $\mathbb{H}$

$\{\gamma_t\}$: shape of interfaces with its occurrence prob. $\mathbf{P}[\{\gamma_t\}]$

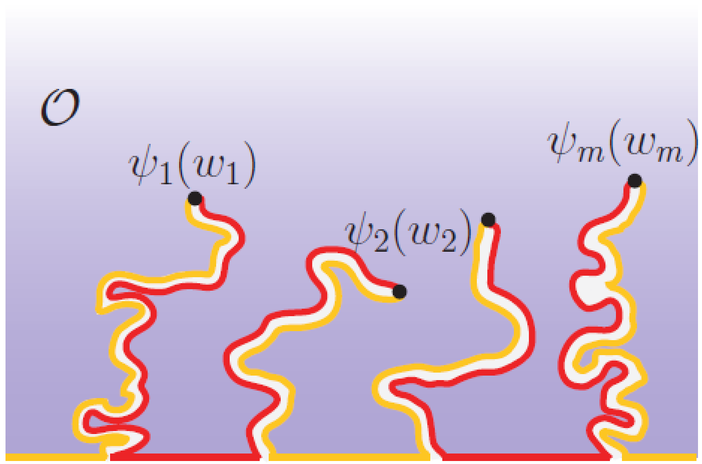
$\langle \mathcal{O} \rangle |_{\{\gamma_t\}}$
thermal average of an observable $\mathcal{O}$ under $\{\gamma_t\}$



- thermal expectation value

$$\langle \mathcal{O} \rangle = \mathbf{E}[\langle \mathcal{O} \rangle |_{\{\gamma_t\}}] = \sum_{\{\gamma_t\}} \mathbf{P}[\{\gamma_t\}] \langle \mathcal{O} \rangle |_{\{\gamma_t\}}$$

$\langle \mathcal{O} \rangle |_{\{\gamma_t\}}$ is time independent (conserved in mean) (SLE martingale)

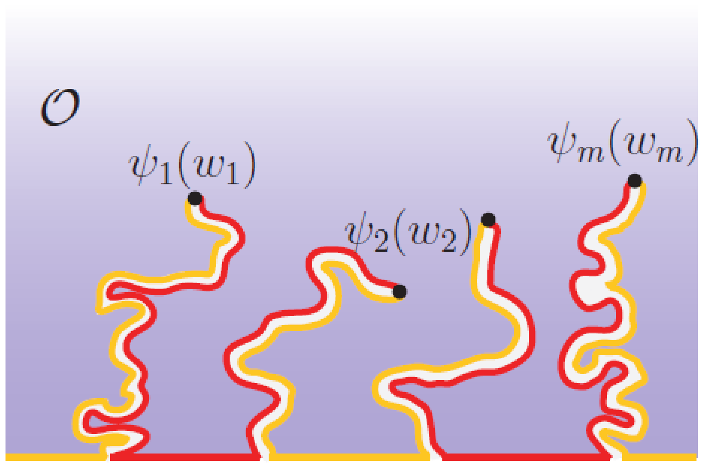


SLE martingale $\mathcal{M}_t := \langle \mathcal{O} \rangle|_{\{\gamma_t\}}$

CFT Correlation function

$$\mathcal{M}_t = \frac{\langle \mathcal{O}\psi_1(w_1)\cdots\psi_m(w_m)\psi_{m+1}(\infty)\rangle}{\langle \psi_1(w_1)\cdots\psi_m(w_m)\psi_{m+1}(\infty)\rangle}$$

$\psi_\alpha(w_\alpha)(=\psi_{\lambda_\alpha}(w_\alpha))$: bcc operators taking their values in rep. specified by the highest weight λ_α



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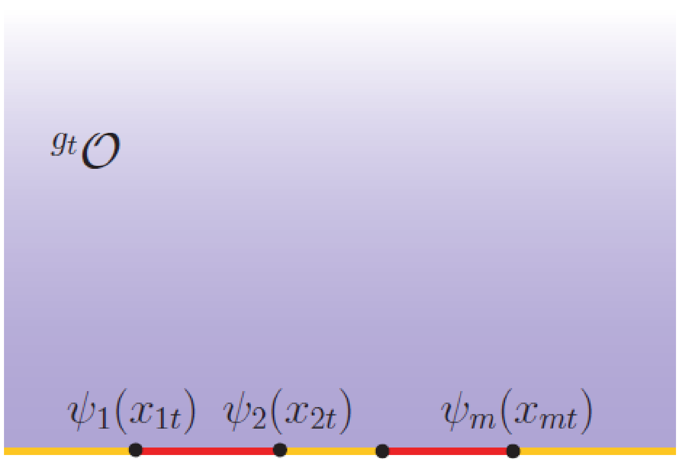
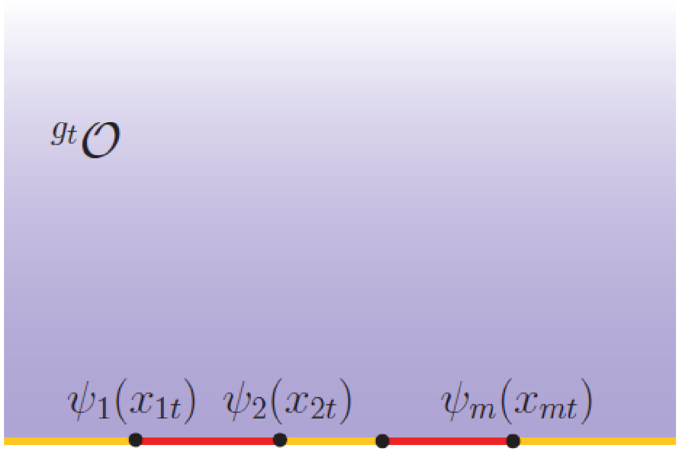
conformal trans. $g_t(z)$



$$\mathcal{M}_t = \frac{\langle g_t\mathcal{O}\psi_1(x_{1t})\cdots\psi_m(x_{mt})\psi_{m+1}(\infty)\rangle}{\langle \psi_1(x_{1t})\cdots\psi_m(x_{mt})\psi_{m+1}(\infty)\rangle}$$

$$dx_{\alpha t} = \underbrace{\sqrt{\kappa}d\xi_{\alpha t}}_{\text{Brownian motion}} + \underbrace{dF_{\alpha t}}_{\text{Drift term}}$$



$$\mathcal{M}_t = \frac{\langle g_t\mathcal{O}\psi_1(x_{1t})\cdots\psi_m(x_{mt})\psi_{m+1}(\infty)\rangle}{\langle\psi_1(x_{1t})\cdots\psi_m(x_{mt})\psi_{m+1}(\infty)\rangle}$$

$$dx_{\alpha t} = \sqrt{\kappa}d\xi_{\alpha t} + dF_{\alpha t}$$

SLE martingale $\mathbf{E}[d\mathcal{M}_t] = 0$ determines the drift terms and the structure of bcc operators.

▪ BCC operators have the null state at level 2.

$$0 = |\chi\rangle := \left(\frac{\kappa}{2}L_{-1}^2 - 2L_{-2} + \frac{\tau}{2}\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a\right)|\psi_\lambda\rangle \qquad |\psi_\lambda\rangle := \lim_{z\to 0}\psi_\lambda(z)|0\rangle$$

Bettelheim et al (05)
Alekseev, Bytsko, Izyurov (11)

$L_n,\ J_n^a$: generators of the Virasoro and affine Lie algebras.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$
$$[J_n^a, J_m^b] = i\sum_c f^{ab}{}_c J^c_{n+m} + nk\delta^{ab}\delta_{n+m,0}$$
$$[L_n, J_m^a] = -mJ^a_{n+m}$$

$$L_0|\psi_\lambda\rangle = h_\lambda|\psi_\lambda\rangle, \qquad L_n|\psi_\lambda\rangle = 0 \ \ (n>0)$$
$$J_0^a|\psi_\lambda\rangle = -t_\lambda^a|\psi_\lambda\rangle, \quad J_n^a|\psi_\lambda\rangle = 0 \ \ (n>0)$$

$$c = \frac{k\dim\mathfrak{g}}{k+h^\vee} \qquad h_\lambda = \frac{(\lambda,\lambda+2\rho)}{2(k+h^\vee)}$$

$\rho = \sum_{i=1}^r \Lambda_i$: Weyl vector

$h^\vee$: dual Coxeter number

- $\widehat{\mathfrak{su}}(2)_k$

For $\widehat{\mathfrak{su}}(2)_k$ case, the above is valid only for the case that the bcc operators carry the spin-1/2 $(\lambda = 1)$.

$$c = \frac{3k}{k+2}, \; h_\lambda = \frac{3}{4(k+2)}$$

$$\boxed{\begin{aligned} &\kappa = 4, \; \tau = 0 \quad (k=1) \\ &\kappa = \frac{4(k+2)}{k+3}, \; \tau = \frac{2}{k+3} \quad (k \geq 2) \end{aligned}}$$

- Drift terms

$$dF_\alpha = \kappa dq_\alpha \partial_{x_\alpha} \log Z_t + 2 \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{m} \frac{dq_\beta}{x_\alpha - x_\beta}$$

$$dG^a_\alpha = \frac{\tau}{Z_t} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{m} \frac{t^a_{\lambda_\beta} Z_t}{x_\beta - x_\alpha} dq_\alpha$$

Z_t : Partition Function

$$Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle$$

Multiple SLE for WZW models

$$dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}, \qquad dx_{\alpha t} = \sqrt{\kappa} d\xi_{\alpha t} + dF_{\alpha t}$$

$$d\theta_t^a(z) = \sum_{\alpha=1}^{m} \frac{dp_{\alpha t}^a}{z - x_{\alpha t}}, \qquad dp_{\alpha t}^a = \sqrt{\tau} d\vartheta_{\alpha t}^a + dG_{\alpha t}^a \quad (1 \leq a \leq \dim(\mathfrak{g})).$$

$\xi_{\alpha t}$: $\mathbb{R}^m$-valued Brownian motion $\qquad \vartheta_{\alpha t}^a$: $\mathbb{R}^{m \dim \mathfrak{g}}$-valued Brownian motion

Drift terms

$$dF_{\alpha t} = \kappa dq_\alpha \partial_{x_{\alpha t}} \log Z_t + 2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{m} \frac{dq_\beta}{x_{\alpha t} - x_{\beta t}}, \qquad dG_{\alpha t}^a = \frac{\tau}{Z_t} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{m} \frac{t_{\lambda_\beta}^a Z_t}{x_{\beta t} - x_{\alpha t}} dq_\alpha$$

Partition function $\quad Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle$

$$\sum_{a=1}^{\dim \mathfrak{g}} \sum_{\alpha=1}^{m+1} t_{\lambda_\alpha}^a Z_t = 0, \qquad \left(\partial_{z_\alpha} - \frac{1}{k + h^\vee} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{m} \frac{t_{\lambda_\alpha}^a t_{\lambda_\beta}^a}{z_\alpha - z_\beta} \right) Z_t = 0.$$

KZ equation

Some Applications for $\widehat{\mathfrak{su}}(2)_k$ case

Consider geometric properties of the SLE interfaces.

$$dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}, \qquad dx_{\alpha t} = \sqrt{\kappa} d\xi_{\alpha t} + dF_{\alpha t}$$

$$dF_\alpha = \kappa dq_\alpha \partial_{x_\alpha} \log Z_t + 2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{m} \frac{dq_\beta}{x_\alpha - x_\beta}$$

Geometric properties are characterized by the correlation Function.

$$Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle$$

$\psi_\alpha(x_\alpha)$ $(\alpha = 1, \ldots, m)$ carries spin-1/2.

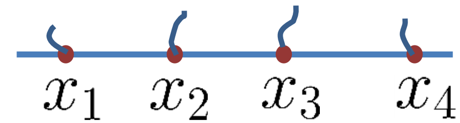
- Fusion rules of the primary operators of the $\widehat{\mathfrak{su}}(2)_k$ WZW models are similar to those in the minimal CFTs.

- A similar treatment developed in the multiple SLE for minimal CFTs (Bauer, Bernard, Kytola ('05)) can be directly applied to $\widehat{\mathfrak{su}}(2)_k$ WZW models.

Conjecture 1 *There exists a one-to-one correspondence between topologically inequivalent configurations of* $\widehat{\mathfrak{su}}(2)_k$ *multiple SLE traces and the partition functions.*

Topologically inequivalent configurations of SLE interfaces

- $x_\alpha \ (1 \leq \alpha \leq m)$: start positions of *m* interfaces

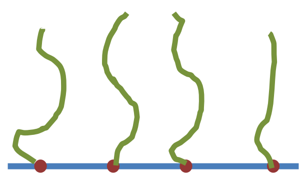


- Interfaces eventually form *m*-*n* disjoint curves. Namely *n* curves form arches and *m*-2*n* curves converge toward ∞.
- Then the number of topologically inequivalent configurations are

$$c_{m,n} = \binom{m}{n} - \binom{m}{n-1}$$

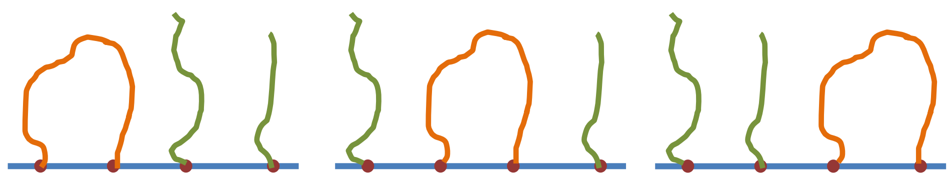
e.g. *m*=4

n=0



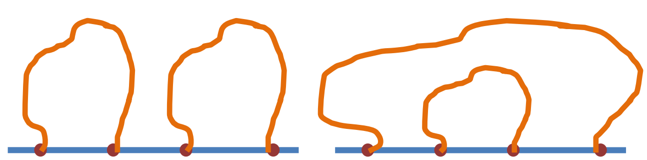
$$c_{4,0} = 1$$

n=1



$$c_{4,1} = 3$$

n=2



$$c_{4,2} = 2$$

- $c_{m,n}$ is a Kostka number appearing as the coefficient of the irreducible decomposition for the *m* tensor product of the $\mathfrak{su}(2)$ fundamental representation $L_{\Lambda}^{\otimes m}$ into $L_{(m-2n)\Lambda}$.

$$L_{\Lambda}^{\otimes m} = \bigoplus_{n=0}^{\lfloor m/2 \rfloor} c_{m,n} L_{m-2n} \qquad c_{m,n} = \binom{m}{n} - \binom{m}{n-1}$$

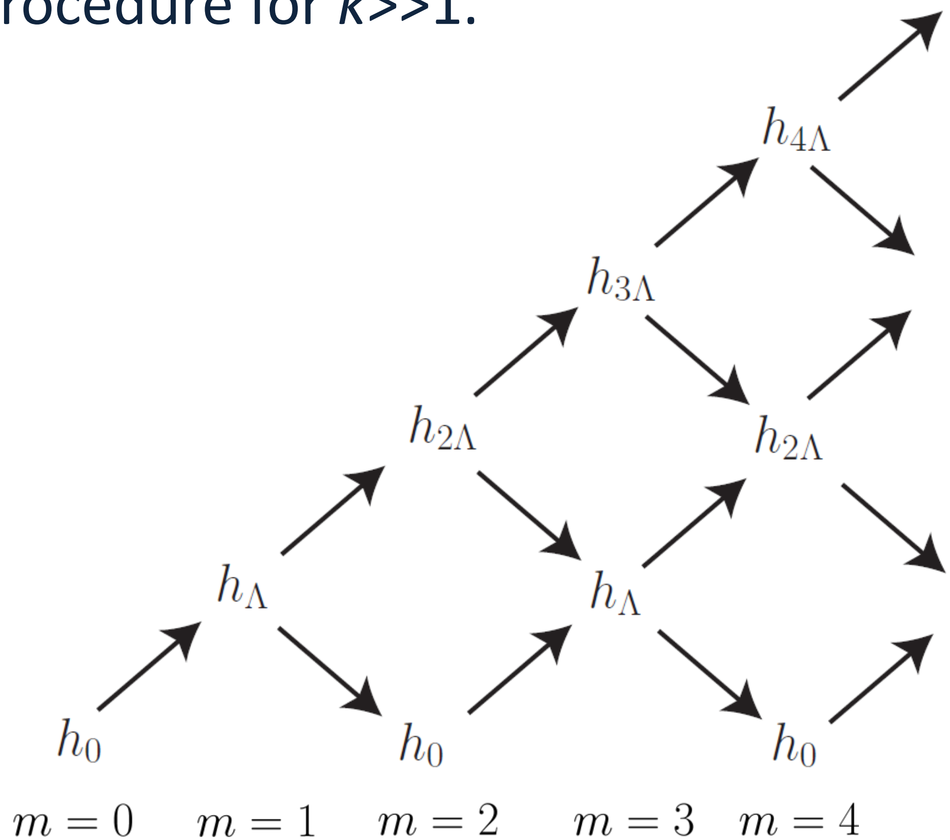
- Applying the fusion procedure to the primary fields constructing the partition function Z_t ,

$$Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle$$

we can classify Z_t in terms of topologically inequivalent configurations of the interfaces.

$$Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle$$

Fusion procedure for *k*>>1.



The number of paths to $h_{(m-n)\Lambda}$ is equivalent to $c_{m,n}$.

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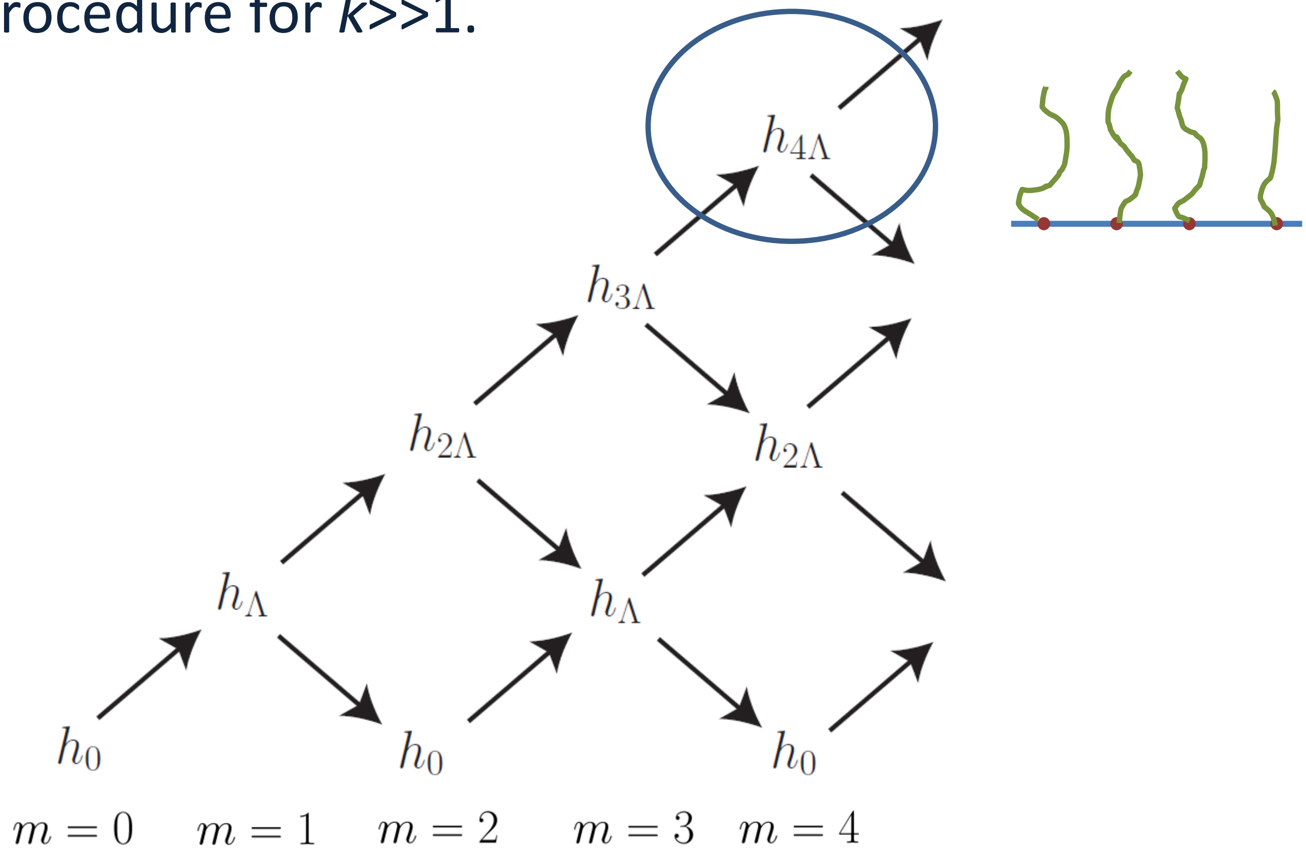
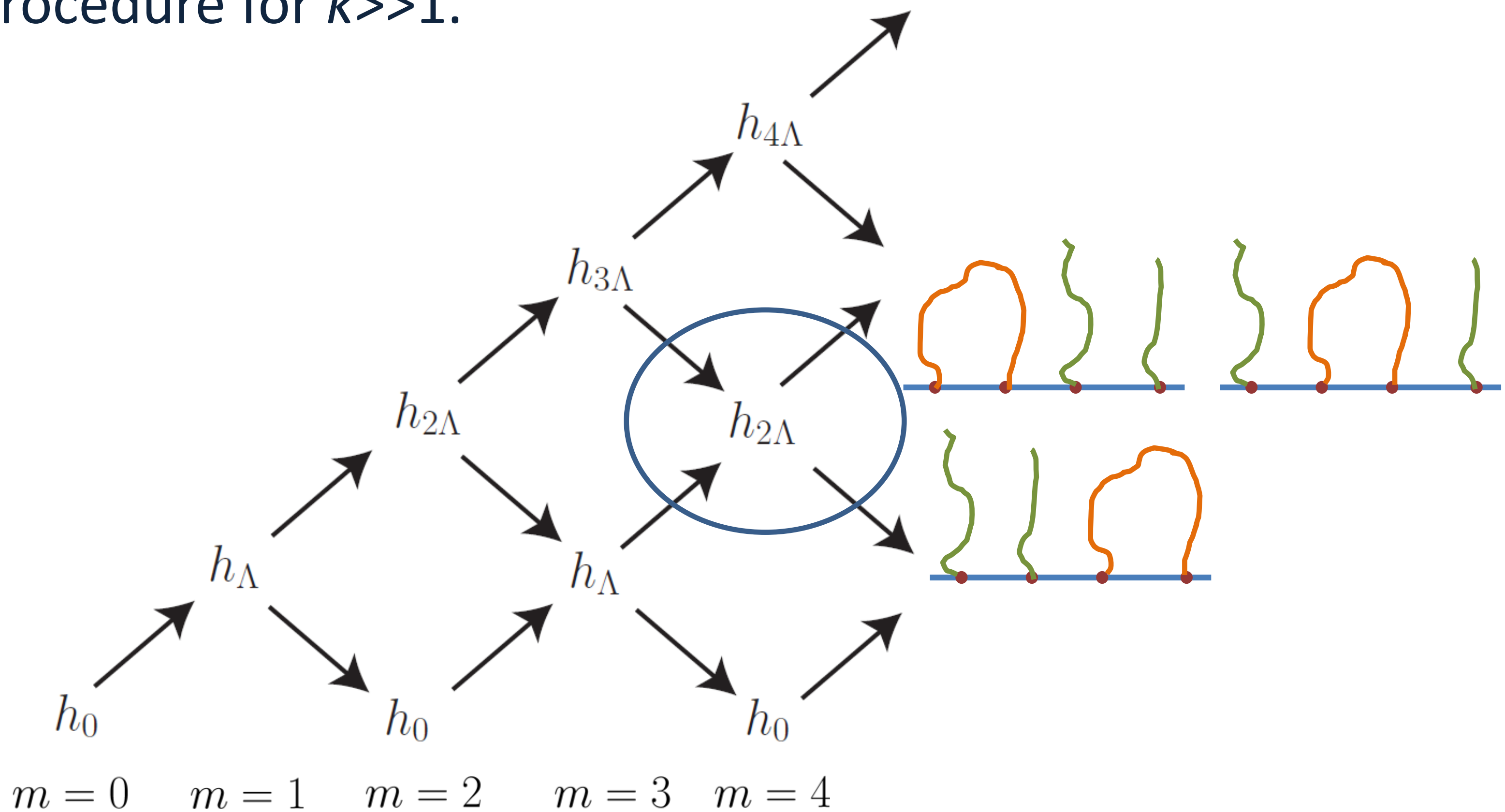


The number of paths to $h_{(m-n)\Lambda}$ is equivalent to $c_{m,n}$.

$$Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle$$

Fusion procedure for *k*>>1.



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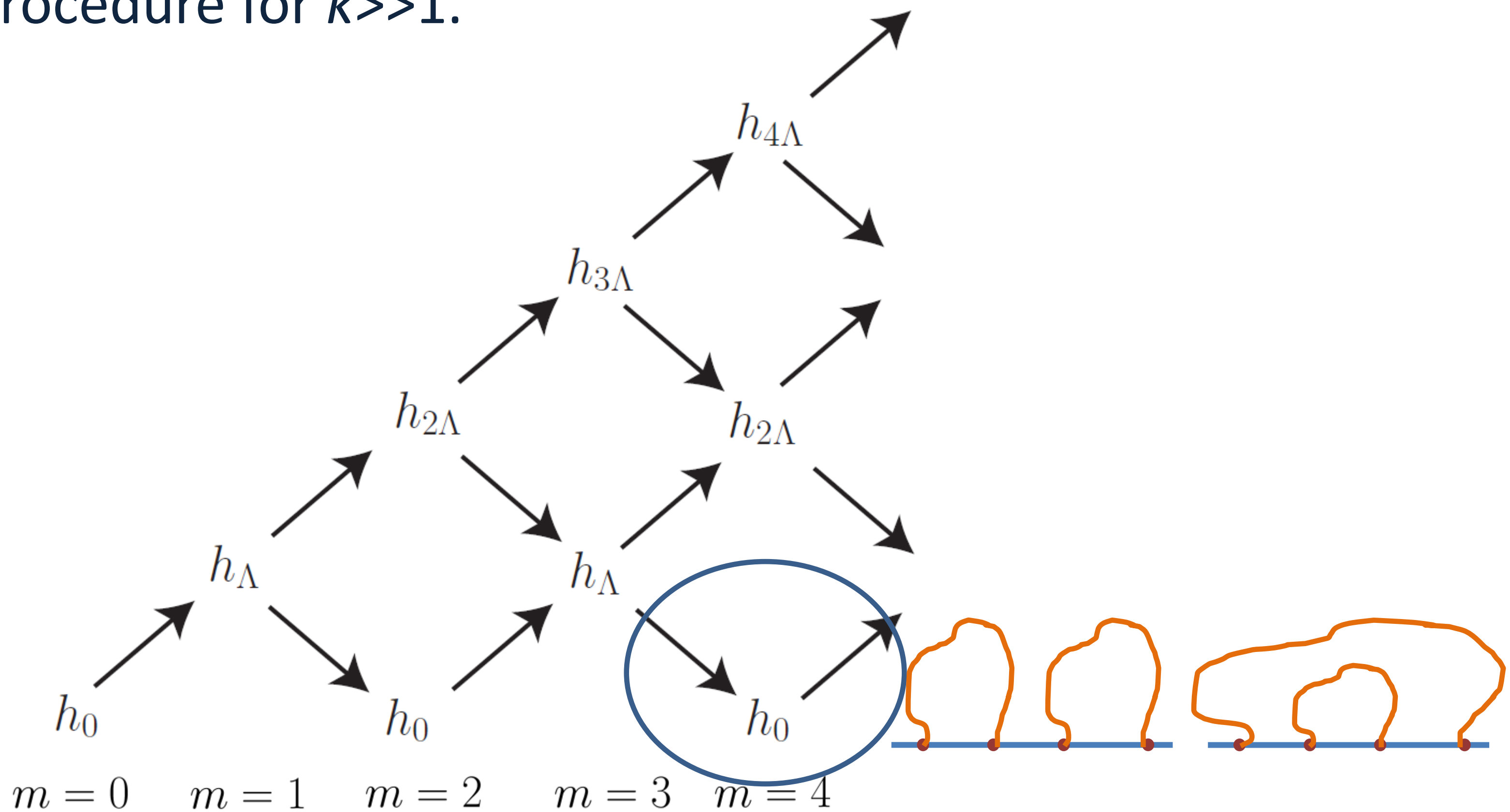
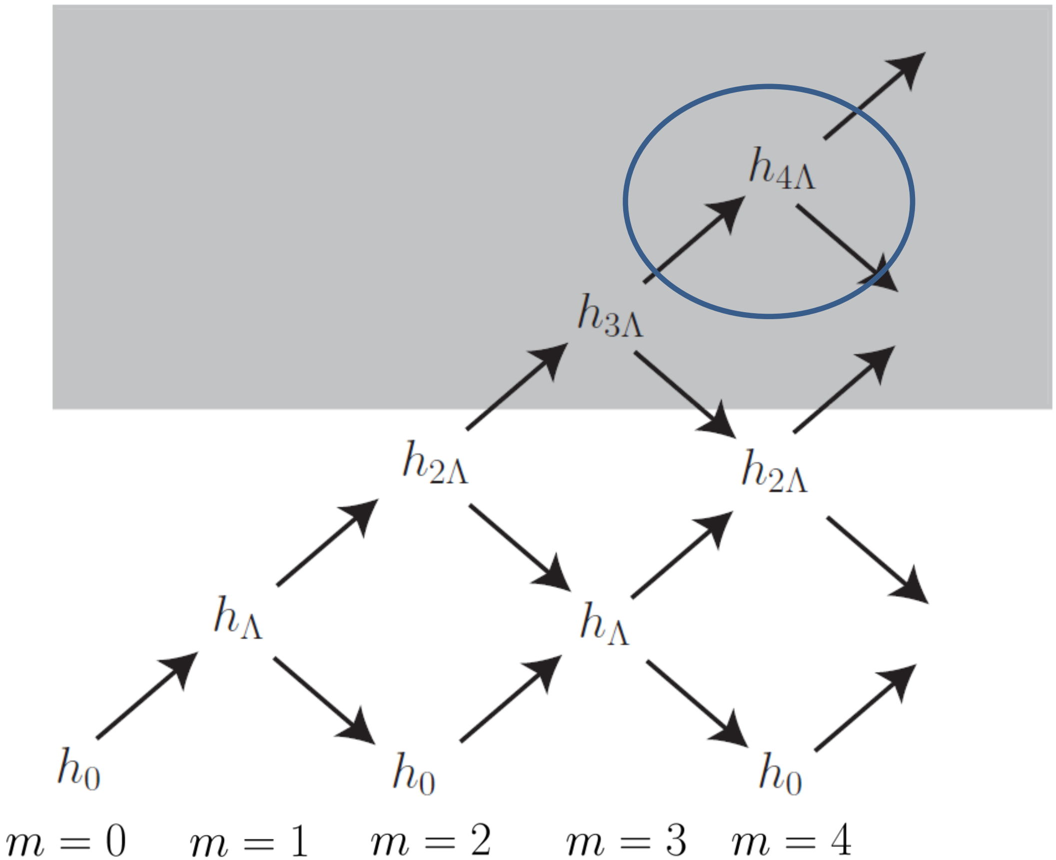
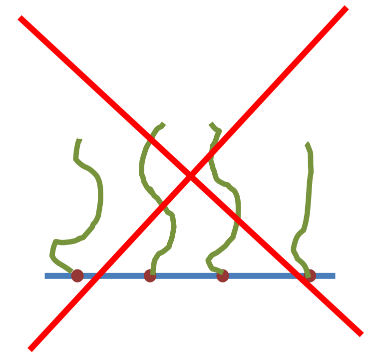


The number of paths to $h_{(m-n)\Lambda}$ is equivalent to $c_{m,n}$.

Fusion procedure for *k*=2.



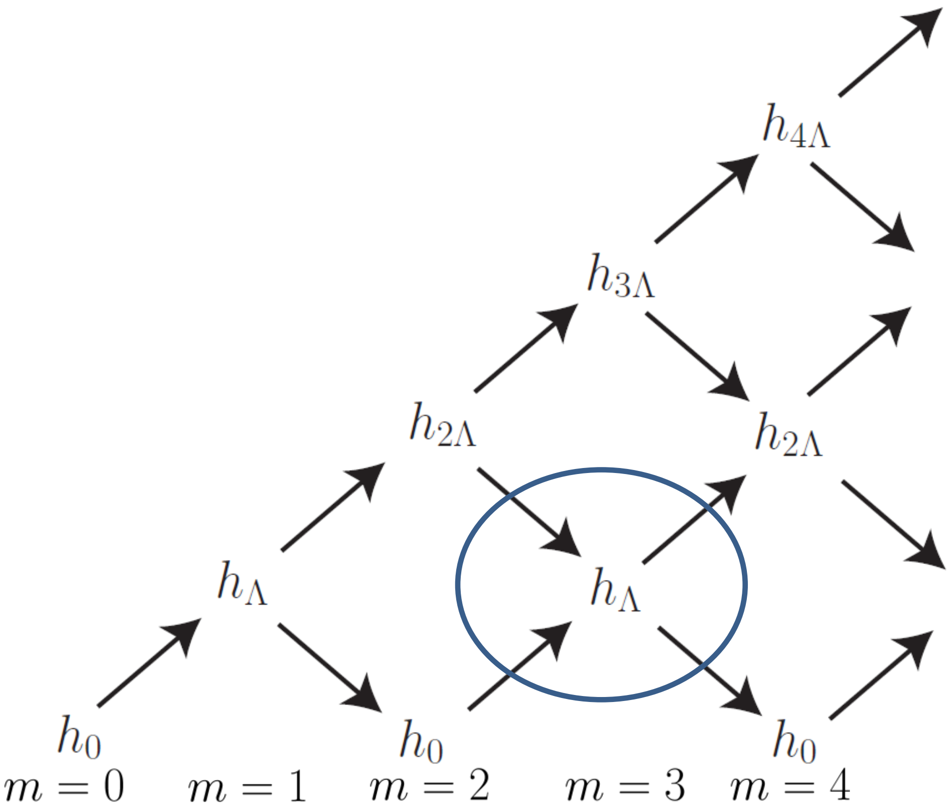


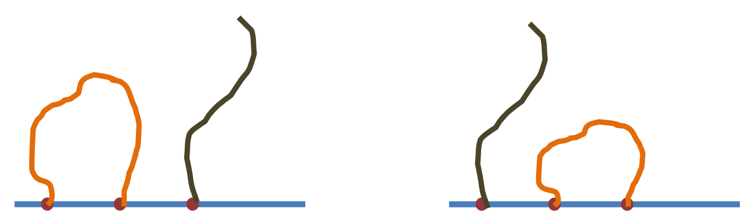
Forbidden!

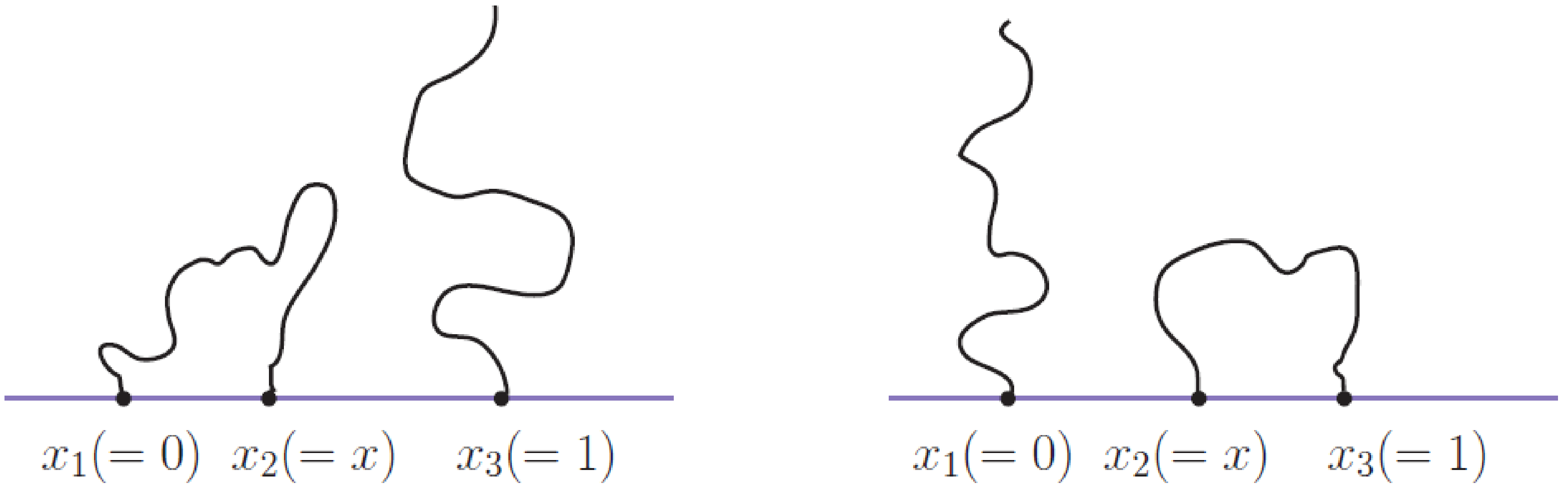
- For generic *k*, the paths are constrained by the fusion rules.

- This conjecture is exactly proved for *m*=2.

- *m*=3 and *n*=1: more interesting. Two topologically inequivalent cofigurations appear. The probabilities for the occurrence of the configuration can be exactly calculated by solving the KZ-eq.







$$\begin{aligned} Z(x) &= \langle \psi_\lambda(\infty)\psi_\Lambda(\infty)\psi_\Lambda(1)\psi_\Lambda(x)\psi_\Lambda(0)\rangle \\ &= \lim_{x\to\infty} x^{2h_\lambda}\langle \psi_\lambda(\infty)\psi_\Lambda(x)\psi_\Lambda(1)\psi_\Lambda(x)\psi_\Lambda(0)\rangle \end{aligned}$$

Solving the KZ-equation, we have

$$Z(x) = Z_{\mathrm{C}_1}(x) + Z_{\mathrm{C}_2}(x)$$

$$Z_{\mathrm{C}_1}(x) = F_1^{(-)} + \frac{1-c_-}{c_+}F_1^{(+)}, \quad Z_{\mathrm{C}_2}(x) = F_2^{(-)} + \frac{1-c_-}{c_+}F_2^{(+)}$$

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$$F_1^{(-)} = x^{-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1\left(\frac{1}{k+2}, \frac{-1}{k+2}; \frac{k}{k+2}; x\right),$$

$$F_1^{(+)} = x^{h_{2\Lambda}-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1\left(\frac{1}{k+2}, \frac{3}{k+2}; \frac{k+4}{k+2}; x\right),$$

$$F_2^{(-)} = \frac{1}{k} x^{1-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1\left(\frac{k+3}{k+2}, \frac{k+1}{k+2}; 2\frac{k+1}{k+2}; x\right),$$

$$F_2^{(+)} = -2x^{h_{2\Lambda}-2h_\Lambda} (1-x)^{h_{2\Lambda}-2h_\Lambda} {}_2F_1\left(\frac{1}{k+2}, \frac{3}{k+2}; \frac{2}{k+2}; x\right),$$

$$c_- = 2\frac{\Gamma(2/(k+2))\Gamma(-2/(k+2))}{\Gamma(1/(k+2))\Gamma(-1/(k+2))}, \quad c_+ = -2\frac{\Gamma^2(2/(k+2))}{\Gamma(3/(k+2))\Gamma(1/(k+2))}.$$

$$Z_{\mathrm{C}_1}(x) = Z_{\mathrm{C}_2}(1-x)$$

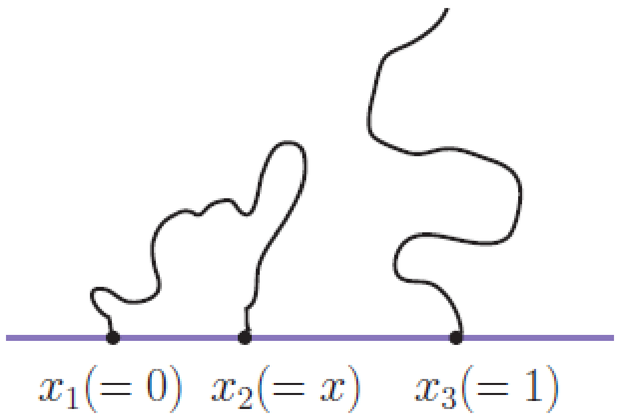
k=1

$$Z_{\mathrm{C}_1}(x) = x^{-1/2}(1-x)^{1/2}, \quad Z_{\mathrm{C}_2}(x) = x^{1/2}(1-x)^{-1/2}.$$

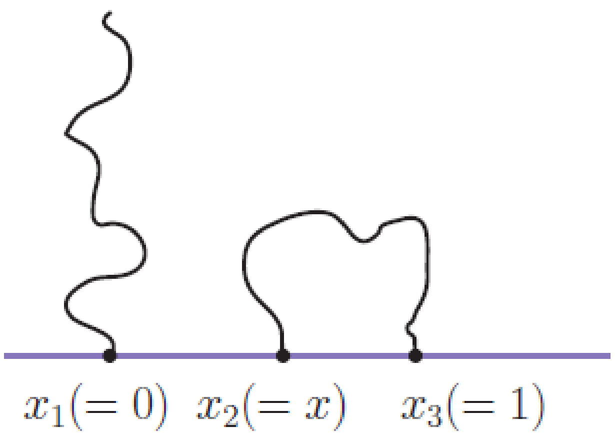
Asymptotics

$$Z_{\mathrm{C}_1}(x) \sim \begin{cases} x^{-2h_\Lambda} & x \to 0 \\ (1-x)^{h_{2\Lambda}-2h_\Lambda} & x \to 1 \end{cases}, \quad Z_{\mathrm{C}_2}(x) \sim \begin{cases} x^{h_{2\Lambda}-2h_\Lambda} & x \to 0 \\ (1-x)^{-2h_\Lambda} & x \to 1 \end{cases}.$$

Z_{C_1}

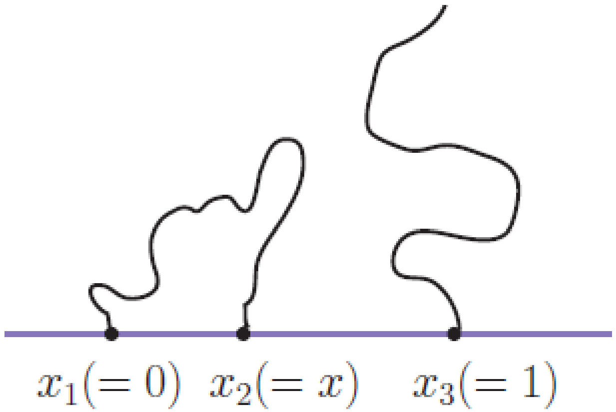


Z_{C_2}



Arch Probabilities

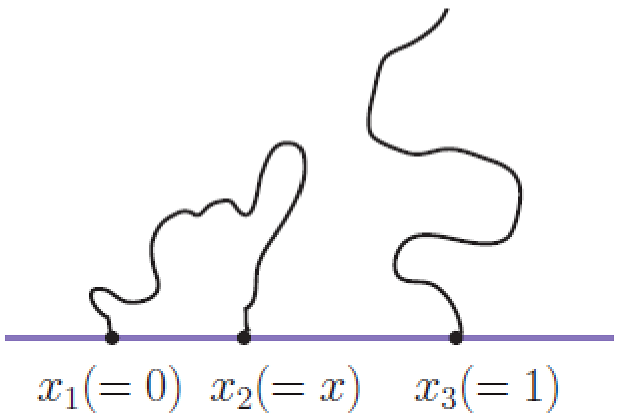
The probability of the occurrence of the configuration



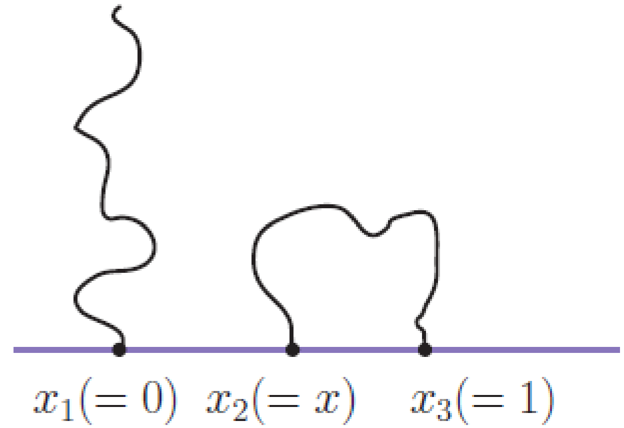
is given by

$$\mathbf{P}[\mathrm{C}_1] = \frac{Z_{\mathrm{C}_1}(x)}{Z_{\mathrm{C}_1}(x) + Z_{\mathrm{C}_2}(x)}$$

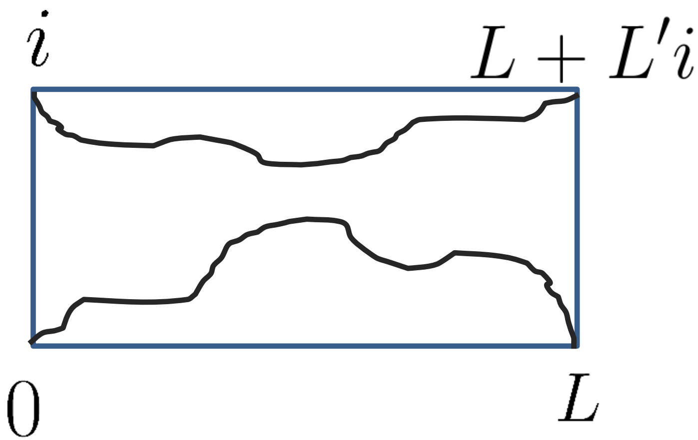
By conformal map (Schwarz-Chiristoffel trans), the configrations

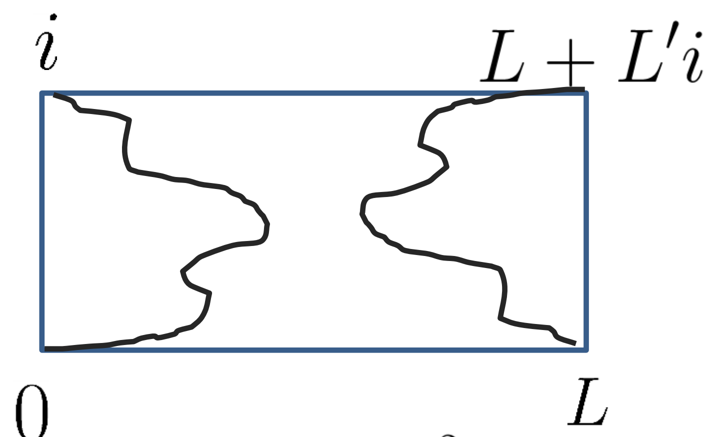


and



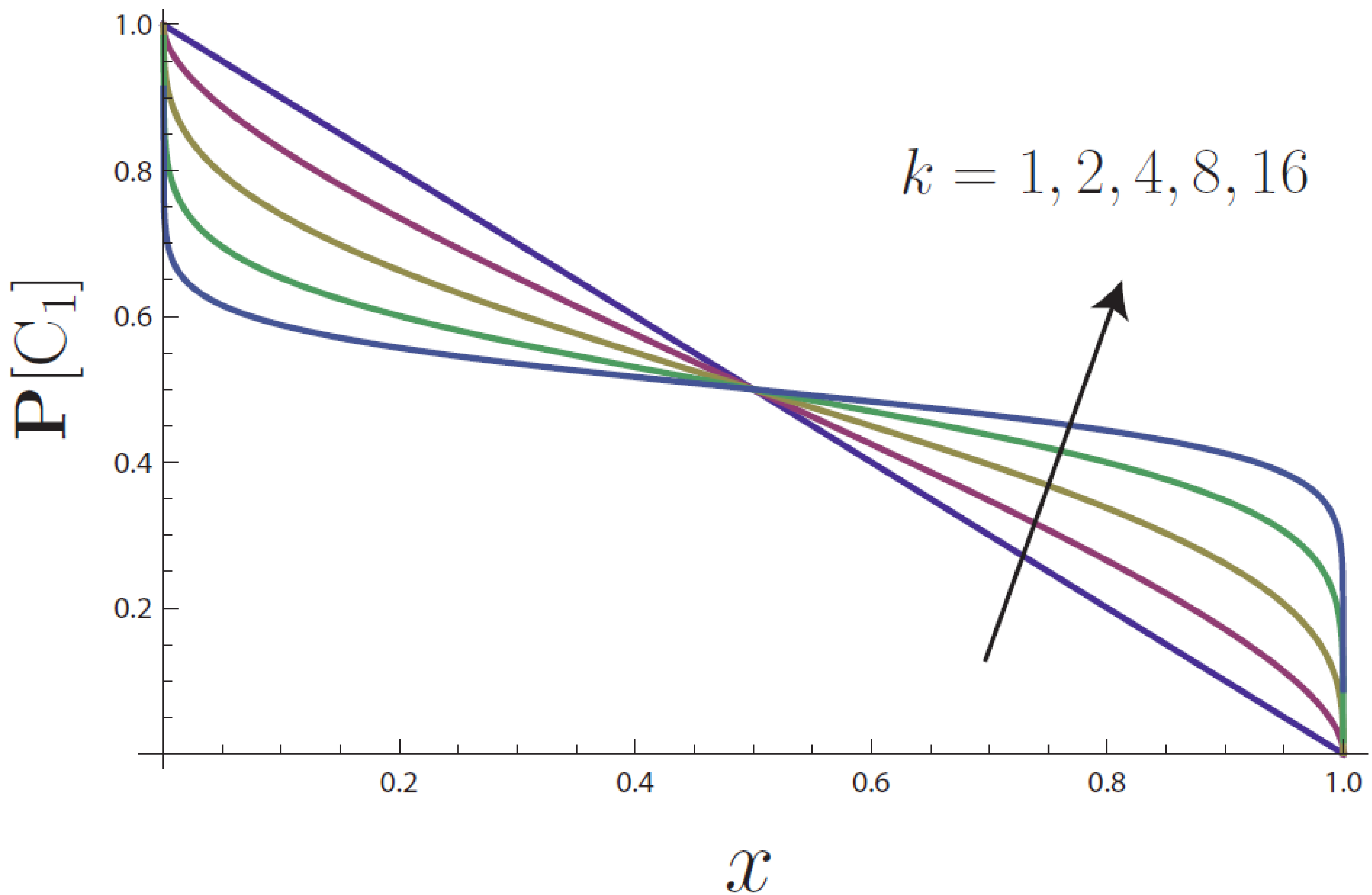
map to





with $r = \frac{L'}{L} = \frac{K(1-l^2)}{2K(l^2)}$ and $x = \frac{(1-l)^2}{(1+l)^2}$

Arch Probabilities



Summary

- We have constructed a multiple version of SLE with extra Lie algebra symmetries.
- The multiple SLE is characterized by the Brownian motion on the Lie group manifold as well as on the real axis .
- As an application, the arch probabilities (crossing probabilities) have been exactly calculated by solving the KZ equation.