Multiple Schramm–Loewner evolutions for conformal field theories with Lie algebra symmetries

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Geometric Aspects of Critical Phenomena

e.g. 2D Ising model

\[ \mathcal{H} = -J \sum_{\langle jk \rangle} \sigma_j \sigma_k \quad (J > 0) \]

\[ T \ll T_c \quad \rightarrow \quad T_c = \frac{2J}{\log(1 + \sqrt{2})} \quad \rightarrow \quad T \gg T_c \]
Critical Phenomena
(Conformal Invariance)

Spin Cluster Boundaries

Random Curves

Fractal Dim. \( d_f = \frac{11}{8} \)

(Saleur and Duplantier (’87))

- Potts model,
- critical percolation, · · ·

Random curves of various models in 2D are directly described by SLE.

2D Ising model \( T = T_c \)
SLE (Schramm-Loewner Evolution)\footnote{Schramm (‘00)}

\[
dg_t(z) = \frac{2dt}{g_t(z) - x_t}, \quad g_0(z) = z \in \mathbb{H}
\]

\[
g_t(z) = z + O(1) \quad (z \to \infty)
\]

\[
x_t = \sqrt{\kappa} \xi_t \quad \xi_t : \text{Standard Brownian Motion}
\]

\[
\mathbb{E}[dx_t] = 0 \text{ and } \mathbb{E}[dx_t dx_t] = \kappa dt
\]

- SLE has a solution up to $\tau_z$. $\tau_z$ : first time when $g_t(z)$ hits $x_t$.
- $g_t^{-1}(x_t)$ defines the tip $\gamma_t$ of the random curve.
- Hull $K_t = \{z \in \mathbb{H} | \tau_z < t\}$. $K_t$ is an increasing family of hulls: $K_s \subset K_t$ for $s < t$. 
defines the tip of the random curve.

Hull $K_t = \{z \in \mathbb{H} | \tau_z < t \}$. $K_t$ is an increasing family of hulls: $K_s \subset K_t$ for $s < t$. 
\[ dg_t(z) = \frac{2dt}{g_t(z) - x_t}, \quad g_0(z) = z \in \mathbb{H} \]
\[ g_t(z) = z + O(1) \quad (z \to \infty) \]
\[ x_t = \sqrt{\kappa} \xi_t \quad \xi_t : \text{Standard Brownian Motion} \]

- \( g_t(z) \) is the unique conformal map uniformizing the complement of the hull \( K_t \).
\[ dg_t(z) = \frac{2dt}{g_t(z) - x_t}, \quad g_0(z) = z \in \mathbb{H} \]

\[ g_t(z) = z + O(1) \quad (z \to \infty) \]

\[ x_t = \sqrt{\kappa} \xi_t \quad \xi_t : \text{Standard Brownian Motion} \]

- The connection between SLE and CFT (conformal field theory) is well understood.  
  Bauer and Bernard (’02)  
  Friedrich and Werner (’02)

\[ c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa} \]

- \( \kappa=6 \) (c=0): critical percolation
- \( \kappa=3 \) (c=1/2): Ising (spin clusters)
- \( \kappa=16/3 \) (c=1/2): Ising (FK clusters)
\[ x_t = \sqrt{\kappa} \xi_t \]

\[ d_f = 1 + \frac{\kappa}{8} \]

\( \kappa = 1 \ (d_f = 9/8) \)

\( \kappa = 2 \ (d_f = 5/4) \)

\( \kappa = 3 \ (d_f = 11/8) \) Ising

\( \kappa = 4 \ (d_f = 3/2) \)
\[ x_t = \sqrt{\kappa} \xi_t \]

\[ d_f = 1 + \kappa/8 \]

\[ \kappa = 5 \quad (d_f = 13/8) \]

\[ \kappa = 6 \quad (d_f = 7/4) \quad \text{critical percolation} \]

\[ \kappa = 7 \quad (d_f = 15/8) \]

\[ \kappa = 8 \quad (d_f = 2) \]
Multiple + Extra Symmetries

- Multiple SLE
  - Multiple random curves
  - Cardy (02)
  - Dubedat (04)
  - Bauer, Bernard, Kytola (05)

- SLE with "spin"
  - Geometric random walks + "algebraic" random walks
  - Bettelheim et al (05)
  - Santachiara (08)
  - Alekseev, Bytsko, Izyurov (11)

Construct the SLE for the WZW models
Derive evolutions so that they are compatible with CFT.

**Geometric part**

\[
dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_\alpha}{gt(z) - x_{\alpha t}}
\]

\[dq_\alpha : \text{time intervals} \quad \sum_\alpha dq_\alpha = dt\]

\[dx_{\alpha t} = \sqrt{\kappa} d\xi_{\alpha t} + dF_{\alpha t} \quad \text{drift term}\]

\[\xi_{\alpha t} : \mathbb{R}^m\text{-valued Brownian motion}\]

\[\mathbb{E}[d\xi_{\alpha t}] = 0, \quad \mathbb{E}[d\xi_{\alpha t}d\xi_{\beta t}] = \delta_{\alpha\beta} dq_\alpha.\]
Derive evolutions so that they are compatible with CFT.

**Geometric part**

\[ dg_t(z) = \sum_{\alpha=1}^{m} \frac{2d\eta_{\alpha}}{g_t(z) - x_{\alpha t}} \]

- \( d\eta_{\alpha} \): time intervals
- \( \sum_{\alpha} d\eta_{\alpha} = dt \)

\[ dx_{\alpha t} = \sqrt{\kappa}d\xi_{\alpha t} + dF_{\alpha t} \]
- drift term

\( \xi_{\alpha t} \): \( \mathbb{R}^m \)-valued Brownian motion

\[ E[d\xi_{\alpha t}] = 0, \quad E[d\xi_{\alpha t}d\xi_{\beta t}] = \delta_{\alpha \beta}d\eta_{\alpha} \]

**algebraic part**

\[ d\theta_{\alpha t}^a(z) = \sum_{\alpha=1}^{m} \frac{dp_{\alpha t}^a}{z - x_{\alpha t}} \]

Stochastic process living on a Lie group manifold

\[ \exp\left[ \sum_a d\theta_{\alpha t}^a(z) t^a \right] \]
- \( t^a \): generators of \( \mathfrak{g} \)
- \( a = 1, \ldots, \dim \mathfrak{g} \)

\[ dp_{\alpha t}^a = \sqrt{\tau}d\vartheta_{\alpha t}^a + dG_{\alpha t}^a \]
- drift term

\( \vartheta_{\alpha t}^a \): \( \mathbb{R}^{m \dim \mathfrak{g}} \)-valued Brownian motion

\[ E[d\vartheta_{\alpha t}^a] = 0, \quad E[d\vartheta_{\alpha t}^a d\vartheta_{\beta t}^b] = \delta^{ab} \delta_{\alpha \beta}d\eta_{\alpha} \]
Derive evolutions so that they are compatible with CFT.

**Geometric part**

\[ dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_{\alpha}}{g_t(z) - x_{\alpha t}} \]

\( dq_{\alpha} : \text{time intervals} \quad \sum_{\alpha} dq_{\alpha} = dt \)

\[ dx_{\alpha t} = \sqrt{\kappa} d\xi_{\alpha t} + dF_{\alpha t} \quad \text{drift term} \]

\( \xi_{\alpha t} : \mathbb{R}^m\text{-valued Brownian motion} \)

\[ \mathbb{E}[d\xi_{\alpha t}] = 0, \quad \mathbb{E}[d\xi_{\alpha t}d\xi_{\beta t}] = \delta_{\alpha\beta} dq_{\alpha} \]

**algebraic part**

\[ d\theta_t^a(z) = \sum_{\alpha=1}^{m} \frac{dp_{\alpha t}^a}{z - x_{\alpha t}} \]

\( dp_{\alpha t}^a = \sqrt{\tau} d\vartheta_{\alpha t}^a + dG_{\alpha t}^a \quad \text{drift term} \)

\( \vartheta_{\alpha t}^a : \mathbb{R}^{m\dim g}\text{-valued Brownian motion} \)

\[ \mathbb{E}[d\vartheta_{\alpha t}^a] = 0, \quad \mathbb{E}[d\vartheta_{\alpha t}^a d\vartheta_{\beta t}^b] = \delta^{ab} \delta_{\alpha\beta} dq_{\alpha} \]

**Stochastic process living on a Lie group manifold**

\( \exp[\sum_a d\theta_t^a(z) t^a] \)

\( t^a : \text{generators of } g \)

\( a = 1, \ldots, \dim g \)

Drift terms are determined by “SLE martingale”.
Consider the corresponding statistical mechanics model on \( \{\gamma_t\} \): shape of interfaces with its occurrence prob. \( P[\{\gamma_t\}] \)

\[
\langle O \rangle \big| \{\gamma_t\} = \text{thermal average of an observable } O \text{ under } \{\gamma_t\}
\]

- thermal expectation value

\[
\langle O \rangle = E[\langle O \rangle \big| \{\gamma_t\}] = \sum_{\{\gamma_t\}} P[\{\gamma_t\}] \langle O \rangle \big| \{\gamma_t\}
\]

\( \langle O \rangle \big| \{\gamma_t\} \) is time independent (conserved in mean) (SLE martingale)
SLE martingale

\[ M_t := \langle O \rangle \big| \{ \gamma_t \} \]

CFT Correlation function

\[ M_t = \frac{\langle O \psi_1(w_1) \cdots \psi_m(w_m) \psi_{m+1}(\infty) \rangle}{\langle \psi_1(w_1) \cdots \psi_m(w_m) \psi_{m+1}(\infty) \rangle} \]

\[ \psi_\alpha(w_\alpha)(= \psi_{\lambda_\alpha}(w_\alpha)) : \text{bcc operators taking their values in rep. specified by the highest weight } \lambda_\alpha \]
SLE martingale
\[ \mathcal{M}_t := \langle \mathcal{O} \rangle \bigg|_{\{\gamma_t\}} \]

CFT Correlation function
\[ \mathcal{M}_t = \frac{\langle \mathcal{O} \psi_1(w_1) \cdots \psi_m(w_m) \psi_{m+1}(\infty) \rangle}{\langle \psi_1(w_1) \cdots \psi_m(w_m) \psi_{m+1}(\infty) \rangle} \]

conformal trans. \( g_t(z) \)

\[ \mathcal{M}_t = \frac{\langle g_t \mathcal{O} \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle}{\langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle} \]

\[ dx_{\alpha t} = \sqrt{\kappa} d\xi_{\alpha t} + dF_{\alpha t} \]

Brownian motion \hspace{1cm} Drift term
SLE martingale $\mathbb{E}[dM_t] = 0$ determines the drift terms and the structure of bcc operators.
BCC operators have the null state at level 2.

\[ 0 = |\chi\rangle := \left( \frac{\kappa}{2} L_{-1}^2 - 2L_{-2} + \frac{T}{2} \sum_{a=1}^{\text{dim } g} J_{-1}^a J_{-1}^a \right) |\psi_\lambda\rangle \quad |\psi_\lambda\rangle := \lim_{z \to 0} \psi_\lambda(z) |0\rangle \]

Bettelheim et al (05)
Alekseev, Bytsko, Izyurov (11)

\( L_n, J_n^a \) : generators of the Virasoro and affine Lie algebras.

\[
\begin{align*}
[L_n, L_m] &= (n - m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} \\
[J_n^a, J_m^b] &= i \sum_c f^{ab}_{\phantom{ab}c} J_c^{n+m} + nk \delta^{ab} \delta_{n+m,0} \\
[L_n, J_m^a] &= -m J_{n+m}^a 
\end{align*}
\]

\[
\begin{align*}
L_0 |\psi_\lambda\rangle &= h_\lambda |\psi_\lambda\rangle, & L_n |\psi_\lambda\rangle &= 0 \ (n > 0) \\
J_0^a |\psi_\lambda\rangle &= -t_\lambda^a |\psi_\lambda\rangle, & J_n^a |\psi_\lambda\rangle &= 0 \ (n > 0)
\end{align*}
\]

\[ \rho = \sum_i r_i \Lambda_i \quad : \text{Weyl vector} \]

\[ \rho = \sum_i r_i \Lambda_i \quad : \text{dual Coxeter number} \]

\[
c = \frac{k \text{dim } g}{k + h^\vee} \\
\rho = \left( \frac{\lambda, \lambda + 2\rho}{2(k + h^\vee)} \right) \\
\rho = \sum_i r_i \Lambda_i \quad : \text{Weyl vector} \\
h^\vee \quad : \text{dual Coxeter number}
\]
For $\hat{su}(2)_k$ case, the above is valid only for the case that the bcc operators carry the spin-$1/2$ ($\lambda = 1$).

\[
c = \frac{3k}{k + 2}, \quad h_\lambda = \frac{3}{4(k + 2)}
\]

\[
\kappa = 4, \quad \tau = 0 \quad (k = 1)
\]

\[
\kappa = \frac{4(k + 2)}{k + 3}, \quad \tau = \frac{2}{k + 3} \quad (k \geq 2)
\]
\( dF_\alpha = \kappa dq_\alpha \partial_{x_\alpha} \log Z_t + 2 \sum_{\substack{\beta = 1 \\ \beta \neq \alpha}}^m \frac{dq_\beta}{x_\alpha - x_\beta} \)

\( dG^a_\alpha = \frac{\tau}{Z_t} \sum_{\substack{\beta = 1 \\ \beta \neq \alpha}}^m \frac{t^a_{\chi_\beta} Z_t}{x_\beta - x_\alpha} dq_\alpha \)

\( Z_t \) : Partition Function

\( Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle \)
Multiple SLE for WZW models

\[ dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}, \quad dx_{\alpha t} = \sqrt{\kappa}d\xi_{\alpha t} + dF_{\alpha t} \]

\[ d\theta^a_t(z) = \sum_{\alpha=1}^{m} \frac{dp_{\alpha t}^a}{z - x_{\alpha t}}, \quad dp_{\alpha t}^a = \sqrt{\tau}d\vartheta_{\alpha t}^a + dG_{\alpha t}^a \quad (1 \leq a \leq \text{dim}(g)). \]

\[ \xi_{\alpha t} : \mathbb{R}^m \text{-valued Brownian motion} \quad \vartheta_{\alpha t}^a : \mathbb{R}^{m \dim g} \text{-valued Brownian motion} \]

**Drift terms**

\[ dF_{\alpha t} = \kappa dq_\alpha \partial x_{\alpha t} \log Z_t + 2 \sum_{\beta=1, \beta \neq \alpha}^{m} \frac{dq_\beta}{x_{\alpha t} - x_{\beta t}} \]

\[ dG_{\alpha t}^a = \frac{\tau}{Z_t} \sum_{\beta=1, \beta \neq \alpha}^{m} \frac{t_{\lambda \beta}^a Z_t}{x_{\beta t} - x_{\alpha t}} dq_\alpha \]

**Partition function**

\[ Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle \]

\[ \sum_{a=1}^{\text{dim } g} \sum_{\alpha=1}^{m+1} t_{\lambda \alpha}^a Z_t = 0, \quad \left( \partial_{z_{\alpha}} - \frac{1}{k + h^\vee} \sum_{a=1}^{\text{dim } g} \sum_{\beta=1, \beta \neq \alpha}^{m} \frac{t_{\lambda \alpha}^a t_{\lambda \beta}^a}{z_{\alpha} - z_{\beta}} \right) Z_t = 0. \]

**KZ equation**
Some Applications for $\text{su}(2)_k$ case

Consider geometric properties of the SLE interfaces.

\[
dg_t(z) = \sum_{\alpha=1}^{m} \frac{2dq_\alpha}{g_t(z) - x_{\alpha t}}, \quad dx_{\alpha t} = \sqrt{\kappa}d\xi_{\alpha t} + dF_{\alpha t}
\]

\[
dF_\alpha = \kappa dq_\alpha \partial x_\alpha \log Z_t + 2 \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{m} \frac{dq_\beta}{x_\alpha - x_\beta}
\]

Geometric properties are characterized by the correlation Function.

\[
Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle
\]

$\psi_\alpha(x_\alpha)$ ($\alpha = 1, \ldots, m$) carries spin-1/2.
• Fusion rules of the primary operators of the $\widehat{su}(2)_k$ WZW models are similar to those in the minimal CFTs.

• A similar treatment developed in the multiple SLE for minimal CFTs (Bauer, Bernard, Kytola (‘05)) can be directly applied to $\widehat{su}(2)_k$ WZW models.

**Conjecture 1** There exists a one-to-one correspondence between topologically inequivalent configurations of $\widehat{su}(2)_k$ multiple SLE traces and the partition functions.
Topologically inequivalent configurations of SLE interfaces

- \( x_\alpha (1 \leq \alpha \leq m) \): start positions of \( m \) interfaces

- Interfaces eventually form \( m-n \) disjoint curves. Namely \( n \) curves form arches and \( m-2n \) curves converge toward \( \infty \).

- Then the number of topologically inequivalent configurations are

\[
c_{m,n} = \binom{m}{n} - \binom{m}{n-1}
\]

e.g. \( m=4 \)

\( n=0 \)
\[
c_{4,0} = 1
\]

\( n=1 \)
\[
c_{4,1} = 3
\]

\( n=2 \)
\[
c_{4,2} = 2
\]
- \( C_{m,n} \) is a Kostka number appearing as the coefficient of the irreducible decomposition for the \( m \) tensor product of the \( \mathfrak{su}(2) \) fundamental representation \( L^\otimes_m \) into \( L_{\Lambda}^{(m-2n)} \).

\[
L^\otimes_m = \bigoplus_{n=0}^{\lfloor m/2 \rfloor} c_{m,n} L_{m-2n}
\]

\[
c_{m,n} = \binom{m}{n} - \binom{m}{n-1}
\]

- Applying the fusion procedure to the primary fields constructing the partition function \( Z_t \),

\[
Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle
\]

we can classify \( Z_t \) in terms of topologically inequivalent configurations of the interfaces.
\[ Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle \]

Fusion procedure for \( k \gg 1 \).

The number of paths to \( h_{(m-n)\Lambda} \) is equivalent to \( c_{m,n} \).
\[ Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle \]

Fusion procedure for \( k \gg 1 \).

The number of paths to \( h_{(m-n)\Lambda} \) is equivalent to \( c_{m,n} \).
\[ Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle \]

Fusion procedure for $k \gg 1$.

The number of paths to $h_{(m-n)\Lambda}$ is equivalent to $c_{m,n}$. 
\[ Z_t = \langle \psi_1(x_{1t}) \cdots \psi_m(x_{mt}) \psi_{m+1}(\infty) \rangle \]

Fusion procedure for \( k \gg 1 \).

The number of paths to \( h_{(m-n)\Lambda} \) is equivalent to \( c_{m,n} \).
Fusion procedure for $k=2$.

- For generic $k$, the paths are constrained by the fusion rules.
This conjecture is exactly proved for $m=2$.

$m=3$ and $n=1$: more interesting. Two topologically inequivalent configurations appear. The probabilities for the occurrence of the configuration can be exactly calculated by solving the KZ-eq.
Solving the KZ-equation, we have

\[ Z(x) = \langle \psi_\lambda(\infty) \psi_\Lambda(\infty) \psi_\Lambda(1) \psi_\Lambda(x) \psi_\Lambda(0) \rangle = \lim_{x \to \infty} x^{2h_\Lambda} \langle \psi_\lambda(\infty) \psi_\Lambda(x) \psi_\Lambda(1) \psi_\Lambda(x) \psi_\Lambda(0) \rangle \]

Solving the KZ-equation, we have

\[ Z(x) = Z_{C_1}(x) + Z_{C_2}(x) \]

\[ Z_{C_1}(x) = F_1^{(-)} + \frac{1 - c_-}{c_+} F_1^{(+)} , \quad Z_{C_2}(x) = F_2^{(-)} + \frac{1 - c_-}{c_+} F_2^{(+)} \]
\[
Z(x) = Z_{C_1}(x) + Z_{C_2}(x)
\]
\[
Z_{C_1}(x) = F_1^{(-)} + \frac{1 - c_-}{c_+} F_1^{(+)}, \quad Z_{C_2}(x) = F_2^{(-)} + \frac{1 - c_-}{c_+} F_2^{(+)}
\]
\[
F_1^{(-)} = x^{-2h_{\Lambda}} (1 - x)^{h_{2\Lambda} - 2h_{\Lambda}} \, _2F_1 \left( \frac{1}{k + 2}, \frac{-1}{k + 2}; \frac{k}{k + 2}; x \right),
\]
\[
F_1^{(+)} = x^{h_{2\Lambda} - 2h_{\Lambda}} (1 - x)^{h_{2\Lambda} - 2h_{\Lambda}} \, _2F_1 \left( \frac{1}{k + 2}, \frac{3}{k + 2}; \frac{k + 4}{k + 2}; x \right),
\]
\[
F_2^{(-)} = \frac{1}{k} x^{1 - 2h_{\Lambda}} (1 - x)^{h_{2\Lambda} - 2h_{\Lambda}} \, _2F_1 \left( \frac{k + 3}{k + 2}, \frac{k + 1}{k + 2}; \frac{2k + 1}{k + 2}; x \right),
\]
\[
F_2^{(+)} = -2x^{h_{2\Lambda} - 2h_{\Lambda}} (1 - x)^{h_{2\Lambda} - 2h_{\Lambda}} \, _2F_1 \left( \frac{1}{k + 2}, \frac{3}{k + 2}; \frac{2}{k + 2}; x \right),
\]
\[
c_- = 2 \frac{\Gamma(2/(k + 2)) \Gamma(-2/(k + 2))}{\Gamma(1/(k + 2)) \Gamma(-1/(k + 2))}, \quad c_+ = -2 \frac{\Gamma^2(2/(k + 2))}{\Gamma(3/(k + 2)) \Gamma(1/(k + 2))}.
\]

\[
k = 1
\]
\[
Z_{C_1}(x) = Z_{C_2}(1 - x)
\]
\[
Z_{C_1}(x) = x^{-1/2} (1 - x)^{1/2}, \quad Z_{C_2}(x) = x^{1/2} (1 - x)^{-1/2}.
\]
Asymptotics

\[ Z_{C_1}(x) \sim \begin{cases} 
  x^{-2h_\Lambda} & x \to 0 \\
  (1 - x)^{h_{2\Lambda} - 2h_\Lambda} & x \to 1
\end{cases}, \quad Z_{C_2}(x) \sim \begin{cases} 
  x^{h_{2\Lambda} - 2h_\Lambda} & x \to 0 \\
  (1 - x)^{-2h_\Lambda} & x \to 1
\end{cases}. \]
Arch Probabilities

The probability of the occurrence of the configuration

is given by

\[ P[C_1] = \frac{Z_{C_1}(x)}{Z_{C_1}(x) + Z_{C_2}(x)} \]
By conformal map (Schwarz-Christoffel trans), the configurations map to

$$i \quad \quad \quad L + L'i$$

with

$$r = \frac{L'}{L} = \frac{K(1 - l^2)}{2K(l^2)}$$

and

$$x = \frac{(1 - l^2)}{(1 + l)^2}$$
Arch Probabilities

$$k = 1, 2, 4, 8, 16$$
Summary

• We have constructed a multiple version of SLE with extra Lie algebra symmetries.
• The multiple SLE is characterized by the Brownian motion on the Lie group manifold as well as on the real axis.
• As an application, the arch probabilities (crossing probabilities) have been exactly calculated by solving the KZ equation.