Classification of Non-Affine Non-Hecke Dynamical R-Matrices

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The G-N-F Dynamical Quantum Yang-Baxter Equation

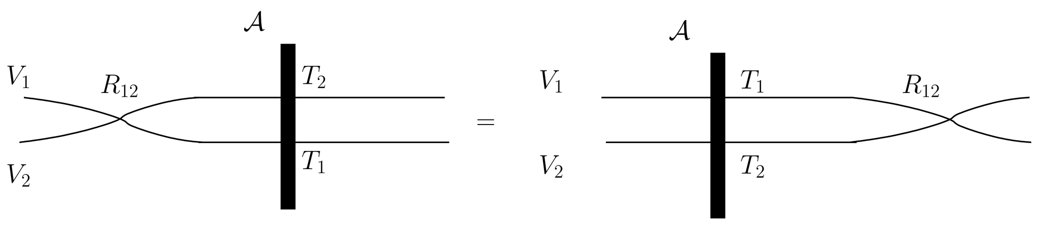
The Gervais-Neveu-Felder dynamical quantum Yang-Baxter equation (G-N-F DQYBE) reads:

$$R_{12}(\lambda+2\gamma h_3)R_{13}(\lambda)R_{23}(\lambda+2\gamma h_1)=R_{23}(\lambda)R_{13}(\lambda+2\gamma h_2)R_{12}(\lambda)$$

This G-N-F DQYBE may be obtained by different approaches:

- Originally formulated by Gervais and Neveu in the context of quantum Liouville theory.
- Built by Felder as a quantization of the so-called modified dynamical classical Yang-Baxter equation. This classical equation itself can be seen as a compatibility condition of Knizhnik-Zamolodchikov-Bernard equations and also arose when considering the Lax formulation of the Calogero-Moser and Ruijsenaar-Schneider model, and particularly its r-matrix.
- Seen as a consistency condition for the dynamical quantum group relation (DQGR): $R_{12}(\lambda+\gamma h_q)T_1(\lambda-\gamma h_2)T_2(\lambda+\gamma h_1)=T_2(\lambda-\gamma h_1)T_1(\lambda+\gamma h_2)R_{12}(\lambda-\gamma h_q)$

The DQRG in short:



T ($T\in\mathrm{End}(V\otimes\mathcal{A})$) is the so-called transfer-matrix and R ($R\in\mathrm{End}(V\otimes V)$) the R-matrix seen as a matrix of structure coefficients for the quadratic exchange relations of $\mathcal{A}$, the considered (dynamical) quantum algebra.

Here, the quantum group relation is dynamical since both R and T depend on a finite family $(\lambda_i)_{i\in\mathbb{N}_n^*}$ of c-number complex "dynamical" parameters , the term "dynamical" comes from the identification of these parameters in the classical limit with the position variables in the context of classical Calogero-Moser or Ruijsenaar-Schneider models.

These "dynamical" parameters λ_i's can be understood as coordinates on the dual algebra $\mathfrak{h}^*$ of a n-dimensional complex Lie algebra $\mathfrak{h}$. We shall suppose here that:

- $\mathfrak{h}$ is abelian
- the finite vector space V is a n-dimensional diagonalizable module of $\mathfrak{h}$

That is: V is a n-dimensional vector space with the weight decomposition $V=\bigoplus_{\mu\in\mathfrak{h}^*}V[\mu]$, where the weight spaces $V[\mu]$ are irreducible modules of $\mathfrak{h}$, hence are one-dimensional.
The operator R is therefore represented by an $n^2\times n^2$ matrix.
The shift of λ (vector of $\mathfrak{h}^*$) by some γh_q is defined for any $v_1,v_2\in V^2$ as follows: $T_1(\lambda+\gamma h_2)v_1\otimes v_2=T_1(\lambda+\gamma\mu_2)v_1\otimes v_2$, where μ_2 is the weight of v_2.

If the R-matrix obeys the so-called zero-weight condition under adjoint action of any element $h\in\mathfrak{h}$: $[h_1+h_2,R_{12}]=0$, the associativity condition on the DQGR implies the G-N-F dynamical quantum Yang-Baxter algebra for R.

Hecke and Weak Hecke conditions

Hecke condition:

The Hecke condition restricts the eigenvalues of the permuted R-matrix $\check{R}=PR$ (P being the permutation operator of vector spaces $V\otimes \mathbb{I}_d$ and $\mathbb{I}_d\otimes V$) to take only:

- one value ϱ on each one-dimensional vector space $V_{ii}=\mathbb{C}v_i\otimes v_i$, for any indice $i\in\mathbb{N}_n^*$,
- and the two distinct values ϱ and $-\kappa$ on each two-dimensional vector space $V_{ij}=\mathbb{C}v_i\otimes v_j\oplus\mathbb{C}v_j\otimes v_i$, for any pair of distinct indices $(i,j)\in(\mathbb{N}_n^*)^2$, $(v_i)_{i\in\mathbb{N}_n^*}$ being a basis of V.

Early examples of non-affine solutions have been brought to light under the hypothesis that R obeys Hecke condition.
The classification of Hecke type non-affine solutions has been succeeded for a long time starting with the pioneering works of Etingof.

weak Hecke condition:

The less constraining"weak Hecke" condition only assumes the existence of two c-numbers ϱ and κ, with $\varrho\neq-\kappa$, such that $(\check{R}-\varrho)(\check{R}+\kappa)=0$.
The weak Hecke condition can be understood as a quantization of the skew-symmetry condition on the classical dynamical r-matrices $r_{12}=-r_{21}$.
Direct connection between classical and quantum dynamical Yang Baxter equation only holds if the r-matrix is skew-symmetric. Dropping the weak Hecke condition may thus modify the understanding of DQYBE as a deformation of a classical structure, but keeps the characteristic quantum structures: coproduct, coactions, fusion of T-matrices and quantum trace (yielding commuting Hamiltonians).
The classification of non-affine R-matrices, solutions of DQYBE, without (weak) Hecke condition was done by Ju, Luo, Wang and Wu in the case of $\mathcal{G}l_2(\mathbb{C})$ and for trigonometric behavior only.

Setting the problem to solve

We propose a complete classification of invertible R-matrices satisfying the zero weight condition solving the DQYBE for $V=\mathbb{C}^n$.
We choose $\mathfrak{h}$ to be the Cartan algebra of $\mathcal{G}l_n(\mathbb{C})$.
This classification is proposed within the following framework:

- We consider non-spectral parameter dependent R-matrices (called "constant" in the litterature on quantum R-matrices but not here to avoid ambiguities with the dependency inthe "dynamical" parameters). This implies a priori no elliptic dependence of the solutions in the dynamical variables,at least in the Hecke case.
- We assume the matrix R to be invertible. Non-invertible R-matrices are expected to correspond to an inadequate choice of auxiliary space V (e.g. reducible). It precludes even the proof of commutation of the traces of monodromy matrices, at least by using the dynamical quantum group structure, hence such R-matrices present a lesser interest.
- We assume that the elements of the matrix R have sufficient regularity properties as functions of their dynamical variables, so that we are able to solve any equation of the form $A(\lambda)B(\lambda)=0$ as $A(\lambda)=0$ or $B(\lambda)=0$ on the whole domain of variation $\mathbb{C}^n$ of λ except of course possible isolated singularities.

The equations

An R-matrix solution of the DQYBE satisfying the zero weight condition takes the form: $R=\sum_{i,j=1}^{n}\Delta_{ij}e_{ij}^{(n)}\otimes e_{ji}^{(n)}+\sum_{i\neq j=1}^{n}d_{ij}e_{ii}^{(n)}\otimes e_{jj}^{(n)}$. The equations obeyed by its coefficients read:

$$\Delta_{ii}\Delta_{ii}(i)\{\Delta_{ii}(i)-\Delta_{ii}\}=0,\quad d_{ij}d_{ij}(i)\{\Delta_{ii}(j)-\Delta_{ii}\}=0,\quad d_{ij}\{\Delta_{ii}(j)\Delta_{ij}(i)-\Delta_{ii}(j)\Delta_{ij}-\Delta_{ji}\Delta_{ij}(i)\}=0,\quad d_{ji}d_{ji}(i)\{\Delta_{ii}(j)-\Delta_{ii}\}=0,\quad d_{ji}\{\Delta_{ii}(j)\Delta_{ij}(i)-\Delta_{ii}(j)\Delta_{ij}-\Delta_{ji}\Delta_{ij}(i)\}=0,$$

$$d_{ij}(i)\{\Delta_{ii}\Delta_{ji}(i)-\Delta_{ii}\Delta_{ji}+\Delta_{ji}\Delta_{ij}(i)\}=0,\quad \Delta_{ii}^2(j)\Delta_{ij}-(d_{ij}d_{ji})\Delta_{ij}(i)-\Delta_{ii}(j)\Delta_{ij}^2=0,\quad d_{ji}(i)\{\Delta_{ii}\Delta_{ji}(i)-\Delta_{ii}\Delta_{ji}+\Delta_{ji}\Delta_{ij}(i)\}=0,\quad \Delta_{ii}^2\Delta_{ji}(i)-(d_{ij}d_{ji})(i)\Delta_{ij}-\Delta_{ii}\Delta_{ji}^2(i)=0,$$

$$\Delta_{ii}d_{ij}(i)d_{ji}(i)-\Delta_{ii}(j)d_{ij}d_{ji}+\Delta_{ij}(i)\Delta_{ji}\{\Delta_{ij}(i)-\Delta_{ji}\}=0,\quad d_{ij}(k)d_{jk}(i)d_{ik}-d_{ij}d_{jk}d_{ik}(j)=0,\quad d_{jk}d_{ik}(j)\{\Delta_{ij}(k)-\Delta_{ij}\}=0,\quad d_{ij}(k)\{\Delta_{ij}(k)\Delta_{jk}+\Delta_{ji}(k)\Delta_{ik}-\Delta_{ik}\Delta_{jk}\}=0,$$

$$d_{ij}(k)d_{ik}\{\Delta_{jk}(i)-\Delta_{jk}\}=0,\quad d_{jk}\{\Delta_{ij}(k)\Delta_{jk}+\Delta_{ik}(j)\Delta_{kj}-\Delta_{ij}(k)\Delta_{ik}(j)\}=0,\quad d_{ij}(k)d_{ji}(k)\Delta_{ik}-d_{jk}d_{kj}\Delta_{ik}(j)+\Delta_{ij}(k)\Delta_{jk}\{\Delta_{ij}(k)-\Delta_{jk}\}=0.,$$

Two equivalence relations yielding the classification

Consistency conditions on the cancelation of d-coefficients and Δ-coefficients lead to relevant partitions and ordering of the indices.
$\mathcal{D}$-relation: The relation defined by $i\mathcal{D}j \Leftrightarrow d_{ij}=0$ is an equivalence relation ($\bar{i}=\mathbb{I}(i)$).
The propagation of the vanishing of Δ-coefficients follows key properties:

- If $\Delta_{ij}=0$, then, $\Delta_{ik}\Delta_{kj}=0$, for any $k\in\mathbb{N}_n^*$.
- If there exist $k\in\mathbb{N}_n^*$ such that $\Delta_{ik}\Delta_{kj}\neq0$, then $\Delta_{ij}\neq0$.

Δ-relation: The relation defined by $i\Delta j \Leftrightarrow \Delta_{ij}\Delta_{ji}\neq0$ is an equivalence relation.

This defines a partition of $\mathbb{N}_n^*=\bigcup_{p=1}^{r}\mathbb{J}_p$ into Δ-classes.

Any $\mathcal{D}$-class $\subset$ one single Δ-class $\Rightarrow$ partition of any Δ-class $J_p=\bigcup_{l=0}^{l_p}\mathbb{I}_l^{(p)}$ in $\mathcal{D}$-classes.

The vanishing of Δ-coefficients is a Δ-class property: For $\mathbb{I},\mathbb{J}$ two distinct Δ-classes, either $\Delta_{ij}=0$ for all $(i,j)\in\mathbb{I}\times\mathbb{J}$, or all $\Delta_{ij}\neq0$. One thus write $\Delta_{\mathbb{I}\mathbb{J}}=0$ or $\Delta_{\mathbb{I}\mathbb{J}}\neq0$.
This leads to introduce a reduced Δ-incidence matrix $M^R\in\mathcal{M}_r(\{0,1\})$, defined by its coefficients $M^R_{\mathbb{I},\mathbb{J}}=1 \Leftrightarrow \Delta_{\mathbb{I}\mathbb{J}}\neq0$ and $M^R_{\mathbb{I},\mathbb{J}}=0 \Leftrightarrow \Delta_{\mathbb{I},\mathbb{J}}=0$.
After partitioning and reordering the indices of Δ-classes, the reduced Δ-incidence matrix M^R is similar to a block diagonal matrix, each diagonal block being upper-triangular (with only 1's in the upper-triangular corner).
This block diagonal decomposition defines a partition of the set $\mathbb{N}_r^*$ of Δ-classes in s subsets $\mathbb{P}_q$ with $q\in\mathbb{N}_s^*$, and an associated partition of $\mathbb{N}_n^*=\bigcup_{q=1}^{s}\mathbb{K}_q$ with $\mathbb{K}_q=\bigcup_{p\in\mathbb{P}_q}\mathbb{J}_p$.

General solution and structure of the set of solutions

Any solution with at least two diagonal blocks in its reduced Δ-incidence matrix $\mathcal{M}^R$, is reducible by some gauge to a decoupled R-matrix (each sub-R-matrix being a solution in a lower dimension) $\Rightarrow$ we will only consider one $\mathbb{K}_q$ block.
To solve the equations and express the solutions let us introduce the functions "sum" $S_{ij}=\Delta_{ij}+\Delta_{ji}$ and "determinant"$\Sigma_{ij}=\begin{vmatrix} d_{ij} & \Delta_{ij} \\ \Delta_{ji} & d_{ji}\end{vmatrix}\neq0$ (since R is invertible).

There is two constants S_q and $\Sigma_q(\neq0)$ and a family of signs $(\epsilon_{\mathbb{I}})_{\mathbb{I}\in\{\mathbb{I}(i),\, i\in\mathbb{K}_q\}}$ such that:

Inside a $\mathcal{D}$-class, $\forall j,j'\in\mathbb{I}(i)$, $\Delta_{jj'}=\Delta_{\mathbb{I}(i)}=(S_q+\epsilon_{\mathbb{I}(i)}D_q)/2$, with $D_q=\sqrt{S_q^2+4\Sigma_q}$.
For indices of distinct $\mathcal{D}$-classes $S_{ij}=S_q$ and $\Sigma_{ij}=\Sigma_q$, $\forall(i,j)\in\mathbb{K}_q^{(2,\not{P})}$.

For indices of distinct $\mathcal{D}$-classes and of the same Δ-class $\mathbb{J}$, there is a family of constants: $(f_{\mathbb{I}})_{\mathbb{I}\in\{\mathbb{I}(i),\, i\in\mathbb{J}\}}\neq0$ such that:

$$\Delta_{ij}((\lambda_k)_{k\in\mathbb{I}(i)\cup\mathbb{I}(j)})=\Delta_{\mathbb{I}(i)\mathbb{I}(j)}(\Lambda_{\mathbb{I}(i)},\Lambda_{\mathbb{I}(j)})=\frac{S_q}{1-\mathbf{e}^{A_q(\epsilon_{\mathbb{I}(i)}\Lambda_{\mathbb{I}(i)}-\epsilon_{\mathbb{I}(j)}\Lambda_{\mathbb{I}(j)})}\frac{f_{\mathbb{I}(i)}}{f_{\mathbb{I}(j)}}}\quad\text{and}$$

$$d_{ij}\,d_{ji}=(B_q-\Delta_{\mathbb{I}(i)\mathbb{I}(j)})\,(B_q-\Delta_{\mathbb{I}(j)\mathbb{I}(i)}),\quad\forall(i,j)\in\mathbb{J}^{(2,\not{P})}\ \text{with}\ e^{A_q}=\frac{D_q-S_q}{D_q+S_q},$$

$B_q=\dfrac{D_q+S_q}{2}=\dfrac{S_q}{1-e^{A_q}}$ and $\Lambda_{\mathbb{I}(i)}=\sum_{k\in\mathbb{I}(i)}\lambda_k$ (in the trigonometric case $S_q\neq0$).
The limit S_q going to 0 yields a rational type of solutions.

For indices of distinct Δ-class $\mathbb{J}_p$ and $\mathbb{J}_{p'}$:

$\Delta_{ij}=S_q=\Delta_{\mathbb{I}(i)\mathbb{I}(j)}$ (thus $S_q\neq0$), $\Delta_{ji}=0$ and $d_{ij}d_{ji}=\Sigma_q$, $\forall(i,j)\in\mathbb{J}_p\times\mathbb{J}_{p'}$.

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