Aspects of the q-deformed $AdS_5 \times S^5$ superstring S-matrix

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(*) based on work in collaboration with:

B. Hoare (London) and T.J. Hollowood (Swansea):

arXiv:1206.0010: “Bound states of the q-deformed $AdS_5 \times S^5$ superstring S-matrix”
arXiv:1112.4485: “q-deformation of the $AdS_5 \times S^5$ superstring S-matrix and its relativistic limit”

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Outline

1. From Pohlmeyer reduction to the interpolating S-matrix
2. The interpolating dressing phase
3. Bound states
4. Conclusions and open questions
From Pohlmeyer reduction to the interpolating S-matrix

**Type IIB string theory in** $AdS_5 \times S^5$

Defined by a GS action on the supercoset $\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$

quantized by explicitly **breaking Lorentz invariance** (light-cone gauge)

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Metsaev-Tseytlin’98

Grigoriev-Tseytlin’08

Mikhailov-SchaferNakemi’08

Delduc-Magro-Vicedo’12

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Classically equivalent, via the Pohlmeyer reduction, to a relativistic (SSSSG) theory defined by the Lagrangian action

$$S = S_{gWZW} \left[ SO(1,4) \times SO(5) \right] - \mu^2 \int d^2 x \text{STr} (\Lambda g - 1 \Lambda g) + \text{fermions}$$

Both theories are classically integrable

Different coupling constants: string coupling $g \leftrightarrow g_{WZW}$ level $k$

Mikhailov’05

Schmidt’11

Different Hamiltonian structures, with coordinated Poisson brackets

$\rightarrow$ Interpolating family of Poisson brackets
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From Pohlmeyer reduction to the interpolating S-matrix

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Lorentz covariant solution of superstring theory based on integrability
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How relativistic and non-relativistic theories can be equivalent?
The equivalence is conjectured to remain in the quantum theory. Lorentz covariant solution of superstring theory based on integrability.

How relativistic and non-relativistic theories can be equivalent?

Compare the quantum solutions of both theories.

Understand the differences between relativistic and non-relativistic quantum integrable theories.
World-sheet and PR S-matrices

- The exact S-matrix for both theories involve a graded tensor product of elementary blocks with symmetry

\[ \text{psu}(2|2) \times \mathbb{R}^3 \rightarrow \text{World-sheet theory} \]

\[ U_q(\text{psu}(2|2) \times \mathbb{R}^2) \rightarrow \text{Pohlmeyer reduced theory} \]
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- Blocks are different limits of the fundamental $R$-matrix of the $q$-deformed Hubbard model with symmetry $U_q(\text{psu}(2|2) \times \mathbb{R}^3)$
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Blocks are different limits of the fundamental $R$-matrix of the q-deformed Hubbard model with symmetry $U_q(\text{psu}(2|2) \ltimes \mathbb{R}^3)$

$R = R(g, q)$ depends on two coupling constants

$R[g, q] \xrightarrow{q \rightarrow 1} \text{world-sheet theory, } g = \text{string coupling}$ $\quad \text{Beisert-Koroteev'08}$

$R[g, q] \xrightarrow{g \rightarrow \infty} \text{(relativistic) PR theory} \quad q = e^{i\pi/k}$ $\quad \text{Hoare-Tseytlin'11}$

$\text{Hoare-Hollowood-JLM'11}$
This suggests the existence of an interpolating S-matrix with:

- Matrix structure completely fixed by the R-matrix of the q-deformed Hubbard model $\rightarrow$ satisfies the YBE

- Overall phase ("dressing phase") fixed by physical requirements
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- Matrix structure completely fixed by the R-matrix of the q-deformed Hubbard model \( \rightarrow \) satisfies the YBE

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Other motivations:

- The spectrum of the interpolating theory is truncated to the symmetric representations of rank \(< k\)

  \( \rightarrow \) regularization of TBA calculations taking \( q \rightarrow 1 \) limit

- The q-deformed Hubbard model itself (Alcaraz-Bariev model)
Basics of $U_q(\mathfrak{psu}(2|2) \rtimes \mathbb{R}^3)$

- Chevalley-Serre generators: $\{E_j, K_j, F_j\}$
  
  
  
  $K_j = q^{H_j}$, $j = 1, 2, 3$

- Centres: $\{U, V\}$

- Extra Serre relations

  $\{[E_1, E_2], [E_3, E_2]\} - (q - 2 + q^{-1})E_2E_1E_3E_2 = g\alpha(1 - V^2U^2)$

  $\{[F_1, F_2], [F_3, F_2]\} - (q - 2 + q^{-1})F_2F_1F_3F_2 = g\alpha^{-1}(V^{-2} - U^{-2})$

- Co-product:

  $\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U^{+\delta_{j,2}} \otimes E_j$

  $\Delta(F_j) = F_j \otimes K_j + U^{-\delta_{j,2}} \otimes F_j$
Fundamental one-particle states

- 4-dimensional “short” multiplet \( \langle 0, 0 \rangle \): \[ \{ |\phi^a\rangle, |\psi^\alpha\rangle \} \quad a, \alpha = 1, 2 \]
  bosonic \( \leftrightarrow \) fermonic

- States labelled by two kinematical variables \( x^\pm \)

\[
q^{-1} \left( \frac{x^+}{x^+} + \frac{1}{x^+} \right) - q \left( \frac{x^-}{x^-} + \frac{1}{x^-} \right) = (q - q^{-1}) \left( \xi + \frac{1}{\xi} \right),
\]
\[
\xi = \frac{-ig(q - q^{-1})}{\sqrt{1 - g^2(q - q^{-1})^2}}, \quad g \text{ real}, \quad q = e^{i\pi/k} \implies 0 \leq \xi \leq 1
\]

- Central charges:
  \( U^2 = q^{-1} \frac{x^+ + \xi}{x^- + \xi} \equiv e^{ip/g} \), \( V^2 = q^{-1} \frac{\xi x^+ + 1}{\xi x^- + 1} \equiv e^{iE/g} \)
Dispersion relation \(\equiv\) shortening (BPS) condition

\[
\left[\frac{kE}{\pi g}\right]_{q^{1/2}}^2 = (1 - \xi^2) \left(1 + \frac{4g^2}{[1/2]_q^2} \sin^2 \left(\frac{p}{2g}\right)\right) \quad [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}
\]

- **String limit** \(q \to 1\) (or \(k \to \infty\))

\[
E^2 = \left(\frac{\pi g}{k}\right)^2 \left[1 + 16g^2 \sin^2 \left(\frac{p}{2g}\right)\right]
\]

- **Relativistic limit** \(g \to \infty\)

\[
E^2 - p^2 = [1/2]_q^2 = \left(2 \cos \frac{\pi}{2k}\right)^{-2}
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- **Relativistic limit** $g \to \infty$

$$
E^2 - p^2 = [1/2]_q^2 = \left( 2 \cos \frac{\pi}{2k} \right)^{-2}
$$

$$
\longrightarrow E = [1/2]_q \cosh \theta, \quad p = [1/2]_q \sinh \theta \quad \longleftarrow
$$
The dispersion relation can be solved in terms of $x(u)$

$$x + \frac{1}{x} + \xi + \frac{1}{\xi} = \frac{1 - \xi^2}{\xi} q^{-2iu} \Rightarrow x^\pm = x \left( u \pm \frac{i}{2} \right)$$

- $q^{-2iu} = e^{2\pi u/k} \Rightarrow u$ takes values in the cylinder $u \sim u + ik$
- $x(u)$ has square root branch points at $q^{-2iu_\pm} = \frac{(1 \mp \xi)^2}{1 - \xi^2}$
  with the two branches distinguished by $|x(u)| \geq 1$
- \( f(x^+, x^-) \) takes values on a 4-fold cover of the cylinder \( u \sim u + ik \)

\[
\mathcal{R}_{\pm 2} : \quad |x^+| < 1, \quad |x^-| < 1
\]

\[
\mathcal{R}_1 : \quad |x^+| < 1, \quad |x^-| > 1
\]

\[
\mathcal{R}_0 : \quad |x^+| > 1, \quad |x^-| > 1
\]

\[
\mathcal{R}_{-1} : \quad |x^+| > 1, \quad |x^-| < 1
\]

\[
\mathcal{R}_m \equiv \mathcal{R}_{m+4n}
\]

- Generalized rapidity torus \( z \sim 2m\omega_1 + 2n\omega_2 \)

  - In the relativistic limit \( g \to \infty, |\omega_1/\omega_2| \to \infty \) and the torus degenerates to the rapidity cylinder \( \theta \sim \theta + 2\pi i \)
The interpolating dressing phase

S-matrix with symmetry $U_q(\text{psu}(2\mid 2) \oplus \text{psu}(2\mid 2) \rtimes \mathbb{R}^3)$

$$S(z_1, z_2) : V_1(z_1) \otimes V_1(z_2) \rightarrow V_1(z_2) \otimes V_1(z_1), \quad V_1 = \langle 0, 0 \rangle^2 \times 2$$

$$S(z_1, z_2) = \frac{U_1^2}{U_2^2} \cdot \frac{Z(z_1, z_2)}{\sigma(z_1, z_2)^2} \hat{R}(z_1, z_2) \otimes_{\text{gr}} \hat{R}(z_1, z_2)$$
The interpolating dressing phase

The interpolating S-matrix

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\[
S(z_1, z_2) = \frac{U_1^2}{U_2^2} \cdot \frac{Z(z_1, z_2)}{\sigma(z_1, z_2)^2} \quad \check{R}(z_1, z_2) \otimes_{\text{gr}} \check{R}(z_1, z_2)
\]

\[\check{R}(z_1, z_2) \otimes_{\text{gr}} \check{R}(z_1, z_2) \text{ has a double pole at } x_1^+ = x_2^- \ldots\]

...but $S(z_1, z_2)$ must have a single pole corresponding to a bound state in the $8 \times 8$-dimensional “short” multiplet $V_2 = \langle 1, 0 \rangle^\times 2$

\[
Z(z_1, z_2) = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \cdot \frac{x_1^- x_2^+ - 1}{x_1^+ x_2^- - 1}
\]

\[
U_i^2 = q^{-1} \frac{x_i^+ + \xi}{x_i^- + \xi} = q \frac{1}{x_i^+} + \xi
\]
The dressing phase: Unitarity and crossing

\[ \sigma(z_1, z_2) \equiv \text{overall phase fixed by physical requirements:} \]

\[ \implies \text{Unitarity} \]
\[ \implies \text{Crossing symmetry} \]
\[ \implies \text{Exact spectrum of bound states} \]

(Review) Vieira-Volin’10

...q-deformed version of the AdS/CFT “dressing factor” …..
The interpolating dressing phase

The dressing phase: Unitarity and crossing

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...q-deformed version of the AdS/CFT “dressing factor”......

\[ S_{ij}^{kl}(z_1, z_2) : |i, z_1 \rangle \otimes |j, z_2 \rangle \rightarrow |k, z_2 \rangle \otimes |l, z_1 \rangle, \quad i, j, k, l = 1, \ldots, 16 \]

\[ (R\text{-matrix}) \text{ Unitarity} \]

\[ S_{ij}^{mn}(z_1, z_2) S_{mn}^{kl}(z_2, z_1) = \delta_i^k \delta_j^l \]
Crossing symmetry

\[ S_{ij}^{kl}(z_1, z_2) = S_{klji}^{ij}(z_2 + \omega_2, z_1) \]

- **Antipode** \( s : x^\pm \to 1/x^\pm \) relates \( \mathcal{R}_0 \to \mathcal{R}_{\pm 2} \) on the rapidity torus

\[
\mathcal{R}_0 \to \mathcal{R}_{\pm 1} \to \mathcal{R}_{\pm 2} \equiv z \to z \pm \omega_2
\]

- **Relativistic S-matrix** \( (g \to \infty) \)
  - Relativistic invariance: \( S_{ij}^{kl}(\theta_1, \theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2) \)
  - **Crossing symmetry**
    \[
    S_{ij}^{kl}(\theta) = S_{ij}^{kl}(i\pi - \theta) \quad \text{or} \quad S_{ij}^{kl}(\theta_1, \theta_2) = S_{ij}^{kl}^{kl}(\theta_2 + i\pi, \theta_1)
    \]
  - **Consistency of the notion of crossing symmetry** requires that
    \[
    z \to z + \omega_2 \quad \longleftrightarrow \quad \theta \to \theta + i\pi \quad \rightarrow \quad \theta = +\pi u/k
    \]
$$\sigma^\gamma(x_1^\pm, x_2^\pm) \sigma(x_1^\pm, x_2^\pm) = \frac{x_2^- + \xi}{x_2^+ + \xi} \cdot \frac{x_1^- - x_2^+}{x_1^- - x_2^-} \cdot \frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

$\sigma^\gamma = \text{analytic continuation on } x_1^\pm \text{ along } \mathcal{R}_0 \to \mathcal{R}_1 \to \mathcal{R}_2$

- Same factorization as in the magnon S-matrix: $\chi(x, y) = -\chi(y, x)$

$$\sigma(x_1^\pm, x_2^\pm) = \exp i \left[ \chi(x_1^+, x_2^+) - \chi(x_1^-, x_2^+) - \chi(x_1^+, x_2^-) + \chi(x_1^-, x_2^-) \right]$$
The interpolating dressing phase

$$\sigma^\gamma(x_1^\pm, x_2^\pm) \sigma(x_1^\pm, x_2^\pm) = \frac{x_2^- + \xi}{x_2^+ + \xi} \cdot \frac{x_1^- - x_2^+}{x_1^- - x_2^-} \cdot \frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

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  $$\sigma(x_1^\pm, x_2^\pm) = \exp i \left[ \chi(x_1^+, x_2^+) - \chi(x_1^-, x_2^+) - \chi(x_1^+, x_2^-) + \chi(x_1^-, x_2^-) \right]$$

- Solution on $\mathcal{R}_{0, 0}$: $|x_1^\pm|, |x_2^\pm| > 1$

$$\chi(x_1, x_2) = i \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z - x_1} \oint_{|z'|=1} \frac{dz'}{2\pi i} \frac{1}{z' - x_2} \log \frac{\Gamma_{q^2}(1 + iu(z) - iu(z'))}{\Gamma_{q^2}(1 - iu(z) + iu(z'))}$$

- $q$-gamma function $\Gamma_{q^2}(1 + x) = \frac{1 - q^{2x}}{1 - q^2} \Gamma_{q^2}(x)$

- No singularities in $\mathcal{R}_{0, 0}$... but branched in $\mathcal{R}_{1, 0}$
A puzzle...

Two theories with the same S-matrix but different reality conditions

- **Magnon**: \((x^\pm)^* = x^\mp\) \[
  U = e^{ip/2g}, \quad V = e^{iE/2g}
\]

- **Mirror**: \((x^\pm)^* = 1/x^\mp\) \[
  U = e^{E/2g}, \quad V = e^{p/2g}
\]

- Obtained by the double Wick rotation \(p \rightarrow -iE, E \rightarrow -ip\)
- Needed to define the TBA

Arutyunov-Frolov’07
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- Both theories have the same relativistic limit

- Different spectrum of bound states

Arutyunov-Frolov'07
In the $q \to 1$ limit, the **magnon theory** has a bound state pole at

$$u_1 - u_2 = -i$$

corresponding to $\langle 1, 0 \rangle \times 2$

and the **mirror theory** has a bound state pole at

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 corresponding to \( \langle 1, 0 \rangle \times 2 \)
and the mirror theory has a bound state pole at
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 corresponding to \( \langle 0, 1 \rangle \times 2 \).

In the relativistic limit $g \to \infty$, the S-matrix has a bound state pole at
\[ \theta = i\pi/k \]
in the physical strip.
In the $q \to 1$ limit, the magnon theory has a bound state pole at
\[ u_1 - u_2 = -i \]
which corresponds to $\langle 1, 0 \rangle \times 2$. And the mirror theory has a bound state pole at
\[ u_1 - u_2 = +i \]
which corresponds to $\langle 0, 1 \rangle \times 2$.

In the relativistic limit $g \to \infty$, the S-matrix has a bound state pole at
\[ \theta = i\pi/k \]
in the physical strip.

Consistency of crossing requires $\theta = \pi u/k$.

How to match the bound state poles in the string ($q \to 1$) and relativistic ($g \to \infty$) limits?
In the $q \rightarrow 1$ limit, the **magnon theory** has a bound state pole at
\[ u_1 - u_2 = -i \]
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and the **mirror theory** has a bound state pole at
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in the **physical strip**

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?? How to match the bound state poles in the string ($q \rightarrow 1$) and relativistic ($g \rightarrow \infty$) limits? 😞

~→ poles moving off and on the “physical strip” as we interpolate??
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How to match the bound state poles in the string ($q \to 1$) and relativistic ($g \to \infty$) limits?

Poles moving off and on the "physical strip" as we interpolate?

In the interpolating (non-relativistic) theory, the nature of the bound states depends on their momentum!!
The interpolating theory is expected to have bound states in the atypical (short) representations

\[\begin{array}{c}
\langle a - 1, 0 \rangle \times 2 \\
\langle 0, a - 1 \rangle \times 2
\end{array}\] or \[\begin{array}{c}
\langle a - 1, 0 \rangle \times 2 \\
\langle 0, a - 1 \rangle \times 2
\end{array}\] for \( a = 1, 2, \ldots \) of \( \mathcal{U}_q(\mathfrak{psu}(2|2) \oplus 2 \ltimes \mathbb{R}^3) \)

- Dispersion relation and central charges

\[
q^{-a} \left( x^+ + \frac{1}{x^+} \right) - q^a \left( x^- + \frac{1}{x^-} \right) = (q^a - q^{-a}) \left( \xi + \frac{1}{\xi} \right)
\]

\[
U^2 = q^{-a} \frac{x^+ + \xi}{x^- + \xi}, \quad V^2 = q^{-a} \frac{\xi x^+ + 1}{\xi x^- + 1}
\]

- Also solved in terms of the map \( x(u) \)

\[
x + \frac{1}{x} + \xi + \frac{1}{\xi} = \frac{1 - \xi^2}{\xi} q^{-2iu} \Rightarrow x^\pm = x \left( u \pm \frac{ia}{2} \right)
\]
The S-matrix has two simple poles

\[ S_{su(2)}(z_1, z_2) = \frac{1}{\sigma(z_1, z_2)^2} \cdot \frac{x_1^+ x_2^-}{x_1^- x_2^+} \cdot \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \cdot \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \]

\[ \rightsquigarrow \text{bound state in } \langle 1, 0 \rangle \times 2 \]
The $S$-matrix has two simple poles

- **$su(2)$ sector:** $|\phi^a \phi^a, z_1\rangle \otimes |\phi^a \phi^a, z_2\rangle \rightarrow |\phi^a \phi^a, z_2\rangle \otimes |\phi^a \phi^a, z_1\rangle$

  $$S_{su(2)}(z_1, z_2) = \frac{1}{\sigma(z_1, z_2)^2} \cdot \frac{x_1^+ x_2^-}{x_1^- x_2^+} \cdot \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \cdot \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

  $\approx$ bound state in $\langle 1, 0\rangle \times 2$

- **$sl(2)$ sector:** $|\psi^\alpha \psi^\alpha, z_1\rangle \otimes |\psi^\alpha \psi^\alpha, z_2\rangle \rightarrow |\psi^\alpha \psi^\alpha, z_2\rangle \otimes |\psi^\alpha \psi^\alpha, z_1\rangle$

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  $\approx$ bound state in $\langle 0, 1\rangle \times 2$
The S-matrix has two simple poles

**\( \mathfrak{su}(2) \) sector:** \( |\phi^a \phi^a, z_1\rangle \otimes |\phi^a \phi^a, z_2\rangle \rightarrow |\phi^a \phi^a, z_2\rangle \otimes |\phi^a \phi^a, z_1\rangle \)

\[
S_{\mathfrak{su}(2)}(z_1, z_2) = \frac{1}{\sigma(z_1, z_2)^2} \cdot \frac{x_1^- x_2^-}{x_1^- x_2^+} \cdot \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} \cdot \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}
\]

\( \leadsto \) bound state in \( \langle 1, 0 \rangle \times 2 \)

**\( \mathfrak{sl}(2) \) sector:** \( |\psi^\alpha \psi^\alpha, z_1\rangle \otimes |\psi^\alpha \psi^\alpha, z_2\rangle \rightarrow |\psi^\alpha \psi^\alpha, z_2\rangle \otimes |\psi^\alpha \psi^\alpha, z_1\rangle \)

\[
S_{\mathfrak{sl}(2)}(z_1, z_2) = \frac{1}{\sigma(z_1, z_2)^2} \cdot \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \cdot \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}
\]

\( \leadsto \) bound state in \( \langle 0, 1 \rangle \times 2 \)

😊 No analogue of the physical sheet!!
Bound state equations

\[(\tilde{E}_1 + \tilde{E}_2, \tilde{p}_1 + \tilde{p}_2)\]

\[(\tilde{E}_1 + is, \tilde{p}_1 + ir)\]  \[(\tilde{E}_2 - is, \tilde{p}_2 - ir)\]

Kinematical conditions

- \(U_{bs} = U_1 U_2\)  \(V_{bs} = V_1 V_2\) \(\iff\) \(E_{bs} = E_1 + E_2,\) \(p_{bs} = p_1 + p_2\)

- \(p_1 = \tilde{p}_1 + ir,\) \(p_2 = \tilde{p}_2 - ir\) such that \(p_{bs} \in \mathbb{R}\) \(E_{bs} \in \mathbb{R} > 0\)

- Relativistic case \(\tilde{p}_1 = \tilde{p}_2\)

- \(r > 0\) Wave function \(\psi(x_1, x_2) = e^{ip_1 x_1 + ip_2 x_2} + S(p_1, p_2)e^{ip_1 x_2 + ip_2 x_1}\)

Reflected wave \(|e^{ip_1 x_2 + ip_2 x_1}| = e^{-r(x_2 - x_1)} \to 0\) for \(x_1 \ll x_2\)
Bound states in the mirror theory

- No solutions for $x_1^+ = x_2^-$
- Two branches for $x_1^- = x_2^+ \implies \text{bound states in } \langle 0, 1 \rangle \times 2$

![Diagram showing mirror kinematics and anomalous threshold](image)
Bound states in the magnon theory

- One branch with $x_1^- = x_2^+$ and three branches with $x_1^+ = x_2^-$

★ Bound states in both $\langle 0, 1 \rangle \times 2$ and $\langle 1, 0 \rangle \times 2$
\begin{itemize}
  \item \(|p_{bs}| < \frac{2\pi g}{k} \longrightarrow \langle 0, 1 \rangle \times 2\) \\
      \hspace{1cm} \rightarrow \text{Soliton branch (disappears in the } k \rightarrow \infty \text{ limit)}
  \item \(|p_{bs}| > \frac{2\pi g}{k} \longrightarrow \langle 1, 0 \rangle \times 2\) \\
      \hspace{1cm} \rightarrow \text{Magnon branch (disappears in the relativistic } g \rightarrow \infty \text{ limit)}
\end{itemize}
• \(|p_{bs}| < \frac{2\pi g}{k}\) \quad \rightarrow \quad \langle 0, 1 \rangle \times 2

\rightarrow \text{Soliton branch (disappears in the } k \rightarrow \infty \text{ limit)}

• \(|p_{bs}| > \frac{2\pi g}{k}\) \quad \rightarrow \quad \langle 1, 0 \rangle \times 2

\rightarrow \text{Magnon branch (disappears in the relativistic } g \rightarrow \infty \text{ limit)}

\begin{equation}
|p_{bs}| = \frac{2\pi g}{k} \quad \rightarrow \quad U_1^2 = U_2^2 = V_1^2 = V_2^2 = q
\end{equation}

⇒ the bound state is marginally unstable for decay into two fundamental particles
Bound states

• \(|p_{bs}| < \frac{2\pi g}{k}\) \rightarrow \langle 0, 1\rangle \times 2

\rightarrow \text{Soliton branch (disappears in the } k \rightarrow \infty \text{ limit)}

• \(|p_{bs}| > \frac{2\pi g}{k}\) \rightarrow \langle 1, 0\rangle \times 2

\rightarrow \text{Magnon branch (disappears in the relativistic } g \rightarrow \infty \text{ limit)}

\star \begin{array}{c}
|p_{bs}| = \frac{2\pi g}{k} \\
\rightarrow \quad U_1^2 = U_2^2 = V_1^2 = V_2^2 = q
\end{array}

\Rightarrow \text{the bound state is marginally unstable for decay into two fundamental particles}

\begin{itemize}
\item The nature of the bound state depends on its momentum!!
\end{itemize}
• $|p_{bs}| < \frac{2\pi g}{k} \quad \rightarrow \quad \langle 0, 1 \rangle \times 2$

\quad \rightarrow \quad \text{Soliton branch (disappears in the } k \rightarrow \infty \text{ limit)}

• $|p_{bs}| > \frac{2\pi g}{k} \quad \rightarrow \quad \langle 1, 0 \rangle \times 2$

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$|p_{bs}| = \frac{2\pi g}{k} \quad \rightarrow \quad U_1^2 = U_2^2 = V_1^2 = V_2^2 = q$

\Rightarrow \quad \text{the bound state is marginally unstable for decay into two fundamental particles}

⚠️ The nature of the bound state depends on its momentum!!

⚠️ In relativistic theories, boosts do not change the intrinsic properties of states!!
Solving the puzzle...

- Bound states in the (string) $k \to \infty$ limit:
  
  Magnon theory $\longrightarrow \langle a-1, 0 \rangle \times 2$
  
  Mirror theory $\longrightarrow \langle 0, a-1 \rangle \times 2$

- Bound states in the (relativistic) $g \to \infty$ limit:
  
  Magnon and mirror theories $\longrightarrow \langle 0, a-1 \rangle \times 2$
Solving the puzzle...

- Bound states in the (string) \( k \to \infty \) limit:
  - Magnon theory \( \rightarrow \langle a - 1, 0 \rangle^2 \times 2 \)
  - Mirror theory \( \rightarrow \langle 0, a - 1 \rangle^2 \times 2 \)

- Bound states in the (relativistic) \( g \to \infty \) limit:
  - Magnon and mirror theories \( \rightarrow \langle 0, a - 1 \rangle^2 \times 2 \)

\[
\begin{align*}
\text{string} & \quad (g, \infty) & \quad \langle a - 1, 0 \rangle^2 \times 2 & \quad \text{magnons} \\
\text{solitons} & \quad (g, k) & \quad \langle 0, a - 1 \rangle^2 \times 2 & \quad \text{mirror magnons} \\
\text{relativistic SSSG} & \quad (\infty, k) & \quad \langle 0, a - 1 \rangle^2 \times 2 & \quad = \quad \text{solitons} \\
\text{mirror string} & \quad (g, k) & \quad (g, \infty)
\end{align*}
\]

double Wick rotation
Conclusions

- q-deformed $AdS_5 \times S^5$ superstring S-matrix and its mirror have been constructed
  - Explicit quantum connection between string and PR theories
  - The nature of bound states depends on their momentum
Conclusions

■ q-deformed $AdS_5 \times S^5$ superstring S-matrix and its mirror have been constructed

❖ Explicit quantum connection between string and PR theories

_tiles The nature of bound states depends on their momentum

✝ Underlying infinite dimensional algebra well understood

Beisert-Galleas-Matsumoto’11
de Leeuw-Matsumoto-Regelskis’11
de Leeuw-Regelskis-Torrielli’11

[→ M. de Leeuw’s talk]
Conclusions

q-deformed $AdS_5 \times S^5$ superstring S-matrix and its mirror have been constructed

Explicit quantum connection between string and PR theories

The nature of bound states depends on their momentum

Open questions

Physical unitarity, bootstrap, physical sheet,…

..........
Conclusions

- q-deformed $AdS_5 \times S^5$ superstring S-matrix and its mirror have been constructed
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- Physical meaning of the interpolating theory in AdS/CFT
Conclusions

- q-deformed $AdS_5 \times S^5$ superstring S-matrix and its mirror have been constructed

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Open questions

- Physical unitarity, bootstrap, physical sheet,…

- Physical meaning of the interpolating theory in AdS/CFT

- Applications in the context of the q-deformed Hubbard model