Aspects of the q-deformed $AdS_5 \times S^5$ superstring S-matrix

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(*) based on work in collaboration with:

B. Hoare (London) and **T.J. Hollowood (Swansea)**:

arXiv:1206.0010: "Bound states of the q-deformed $AdS_5 \times S^5$ superstring S-matrix"

arXiv:1112.4485: "q-deformation of the $AdS_5 \times S^5$ superstring S-matrix and its relativistic limit"

arXiv:1107.0628: "A relativistic relative of the magnon S-matrix"

RAQIS'12 Angers, September 2012

Outline

Type IIB string theory in $AdS_5 \times S^5$

Metsaev-Tseytlin'98

☞ Defined by a GS action on the supercoset $\boxed{\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}}$

quantized by explicitly breaking Lorentz invariance (light-cone gauge)

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Grigoriev-Tseytlin'08
Mikhailov-SchaferNakemi'08
Delduc-Magro-Vicedo'12

- **Classically** equivalent, via the Pohlmeyer reduction, to a relativistic (SSSSG) theory defined by the Lagrangian action

$$\boxed{S = S_{\text{gWZW}}\left[\frac{SO(1,4)\times SO(5)}{SU(2)^{\times 4}}\right] - \mu^2 \int d^2x\, \text{STr}\left(\Lambda\gamma^{-1}\Lambda\gamma\right) + \text{fermions}}$$

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  - Both theories are classicaly integrable
  - Different coupling constants: string coupling $g \longleftrightarrow$ gWZW level k

Mikhailov'05
Schmidtt'11

  - Different Hamiltonian structures, with coordinated Poisson brackets

  $\longrightarrow$ Interpolating family of Poisson brackets

Grigoriev-Tseytlin'08

★ The equivalence is conjectured to remain in the quantum theory

$\xrightarrow{??}$ Lorentz covariant solution of superstring theory based on integrability

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How relativistic and non-relativistic theories can be equivalent?

Compare the quantum solutions of both theories

Understand the differences between relativistic and non-relativistic quantum integrable theories

World-sheet and PR S-matrices

☞ The exact S-matrix for both theories involve a graded tensor product of elementary blocks with symmetry

- $\boxed{\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3} \longrightarrow$ World-sheet theory
- $\boxed{U_q(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2)} \longrightarrow$ Pohlmeyer reduced theory

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- Blocks are different limits of the fundamental R-matrix of the q-deformed Hubbard model with symmetry $\boxed{U_q\big(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3\big)}$

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★ $R = R(g, q)$ depends on two coupling constants

Beisert-Koroteev'08

$$R[g,q] \xrightarrow{q \to 1} \text{world-sheet theory}, \quad g = \text{string coupling}$$

Hoare-Tseytlin'11
Hoare-Hollowood-JLM'11

$$R[g,q] \xrightarrow{g \to \infty} \text{(relativistic) PR theory} \quad \boxed{q = e^{i\pi/k}}$$

The interpolating S-matrix

☞ This suggests the existence of an interpolating S-matrix with

Beisert-Koroteev'08

- ↦ Matrix structure completely fixed by the R-matrix of the q-deformed Hubbard model ⟶ satisfies the YBE

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- ↦ Overall phase ("dressing phase") fixed by physical requirements

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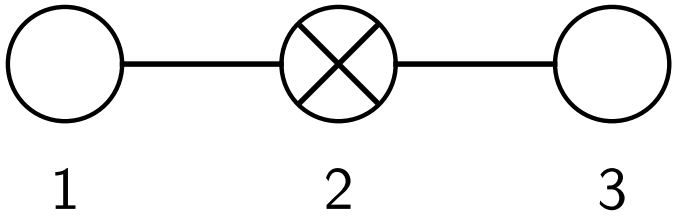
- Other motivations:
  - The spectrum of the interpolating theory is truncated to the symmetric representations of rank $< k$

    Arutyunov-de Leeuw-van Tongeren'12

    ⟶ regularization of TBA calculations taking $q \to 1$ limit
  - The q-deformed Hubbard model itself (Alcaraz-Bariev model)

Basics of $U_q(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$



$$A = \begin{pmatrix} +2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix}$$

- Chevalley-Serre generators: $\boxed{\{E_j, K_j, F_j\}}$ $\quad K_j = q^{H_j}, \quad j = 1, 2, 3$
- Centres: $\boxed{\{U, V\}}$
- Extra Serre relations

$$\{[E_1, E_2], [E_3, E_2]\} - (q - 2 + q^{-1}) E_2 E_1 E_3 E_2 = g\alpha(1 - V^2 U^2)$$

$$\{[F_1, F_2], [F_3, F_2]\} - (q - 2 + q^{-1}) F_2 F_1 F_3 F_2 = g\alpha^{-1}(V^{-2} - U^{-2})$$

- Co-product:

$$\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U^{+\delta_{j,2}} \otimes E_j, \qquad \Delta(F_j) = F_j \otimes K_j + U^{-\delta_{j,2}} \otimes F_j$$

Fundamental one-particle states

- 4-dimensional "short" multiplet $\langle 0,0\rangle$: $\boxed{\{|\phi^a\rangle, |\psi^\alpha\rangle\}}$ $\quad a,\alpha = 1,2$

bosonic $\hookleftarrow$ $\quad\hookrightarrow$ fermionic

Beisert-Koroteev'08
Beisert-Galleas-Matsumoto'11
Hoare-Hollowood-JLM'11

- States labelled by two kinematical variables $x^\pm$

$$q^{-1}\left(x^+ + \frac{1}{x^+}\right) - q\left(x^- + \frac{1}{x^-}\right) = (q - q^{-1})\left(\xi + \frac{1}{\xi}\right),$$

$$\xi = \frac{-ig(q-q^{-1})}{\sqrt{1-g^2(q-q^{-1})^2}}, \qquad g \text{ real},\ q = e^{i\pi/k} \ \Rightarrow\ 0 \le \xi \le 1$$

- Central charges: $U^2 = q^{-1}\frac{x^+ + \xi}{x^- + \xi} \equiv e^{ip/g}, \quad V^2 = q^{-1}\frac{\xi x^+ + 1}{\xi x^- + 1} \equiv e^{iE/g}$

☞ Dispersion relation $\equiv$ shortening (BPS) condition

$$\boxed{\left[\frac{kE}{\pi g}\right]^2_{q^{1/2}} = (1-\xi^2)\left(1+\frac{4g^2}{[1/2]_q^2}\sin^2\left(\frac{p}{2g}\right)\right)} \qquad [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$$

- String limit $q \to 1$ (or $k \to \infty$)

$$E^2 = \left(\frac{\pi g}{k}\right)^2 \left[1 + 16g^2 \sin^2 \frac{p}{2g}\right]$$

- Relativistic limit $g \to \infty$

$$E^2 - p^2 = [1/2]_q^2 = \left(2\cos\frac{\pi}{2k}\right)^{-2}$$

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$$E^2 - p^2 = [1/2]_q^2 = \left(2\cos\frac{\pi}{2k}\right)^{-2}$$

$$\longrightarrow \; E = [1/2]_q \cosh\theta\,, \qquad p = [1/2]_q \sinh\theta \; \longleftarrow$$

☞ The dispersion relation can be solved in terms of $x(u)$

$$x + \frac{1}{x} + \xi + \frac{1}{\xi} = \frac{1-\xi^2}{\xi}\, q^{-2iu} \;\Rightarrow\; x^{\pm} = x\left(u \pm \frac{i}{2}\right)$$

- $q^{-2iu} = e^{2\pi u/k} \;\Rightarrow\;$ u takes values in the cylinder $u \sim u + ik$
- $x(u)$ has square root branch points at $q^{-2iu_\pm} = \frac{(1\mp\xi)^2}{1-\xi^2}$ with the two branches distinguished by $|x(u)| \gtrless 1$

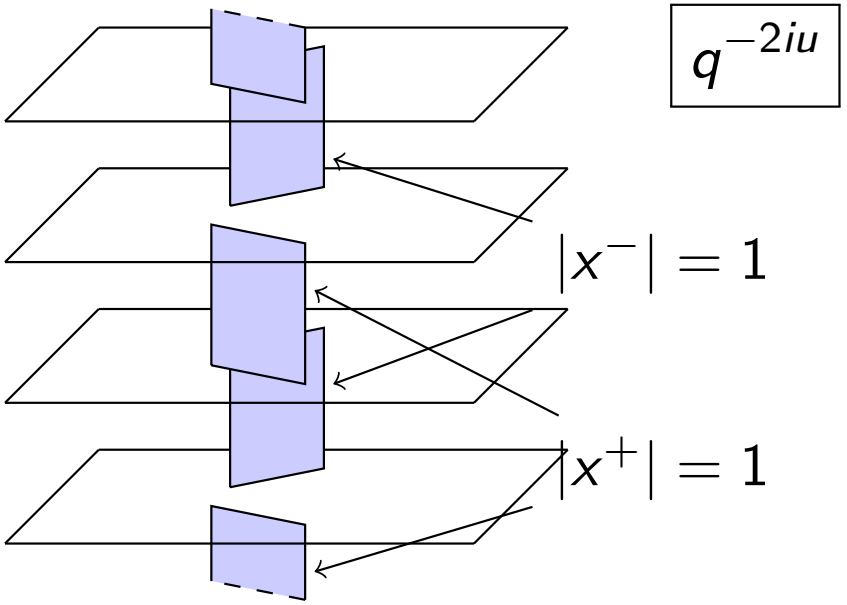
- $f(x^+, x^-)$ takes values on a 4-fold cover of the cylinder $u \sim u + ik$

$$\mathcal{R}_{\pm 2} : \quad |x^+| < 1 \ , \quad |x^-| < 1$$

$$\mathcal{R}_{1} : \quad |x^+| < 1 \ , \quad |x^-| > 1$$

$$\mathcal{R}_{0} : \quad |x^+| > 1 \ , \quad |x^-| > 1$$

$$\mathcal{R}_{-1} : \quad |x^+| > 1 \ , \quad |x^-| < 1$$



$$\boxed{\mathcal{R}_m \equiv \mathcal{R}_{m+4n}}$$

☞ Generalized rapidity torus $z \sim 2m\omega_1 + 2n\omega_2$

- In the relativistic limit $g \to \infty$, $|\omega_1/\omega_2| \to \infty$ and the torus degenerates to the rapidity cylinder $\theta \sim \theta + 2\pi i$

The interpolating S-matrix

- S-matrix with symmetry $U_q(\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$

$$S(z_1, z_2): \; V_1(z_1) \otimes V_1(z_2) \to V_1(z_2) \otimes V_1(z_1), \qquad V_1 = \langle 0, 0 \rangle^{\times 2}$$

$$\boxed{S(z_1, z_2) = \frac{U_1^2}{U_2^2} \cdot \frac{Z(z_1, z_2)}{\sigma(z_1, z_2)^2} \; \check{R}(z_1, z_2) \otimes_{\mathrm{gr}} \check{R}(z_1, z_2)}$$

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☞ $\check{R}(z_1, z_2) \otimes_{\mathrm{gr}} \check{R}(z_1, z_2)$ has a double pole at $x_1^+ = x_2^-$...

...but $S(z_1, z_2)$ must have a single pole corresponding to a bound state in the 8×8-dimensional "short" multiplet $V_2 = \langle 1, 0 \rangle^{\times 2}$

$$\boxed{Z(z_1, z_2) = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \cdot \frac{x_1^- x_2^+ - 1}{x_1^+ x_2^- - 1}} \qquad U_i^2 = q^{-1} \frac{x_i^+ + \xi}{x_i^- + \xi} = q \frac{\frac{1}{x_i^-} + \xi}{\frac{1}{x_i^+} + \xi}$$

The dressing phase: Unitarity and crossing

☞ $\boxed{\sigma(z_1, z_2)} \equiv$ overall phase fixed by physical requirements:

- $\mapsto$ Unitarity
- $\mapsto$ Crossing symmetry
- $\mapsto$ Exact spectrum of bound states

(Review) Vieira-Volin'10

...q-deformed version of the AdS/CFT "dressing factor".....

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- S-matrix amplitudes

$$S_{ij}^{kl}(z_1, z_2) : |i, z_1\rangle \otimes |j, z_2\rangle \longrightarrow |k, z_2\rangle \otimes |l, z_1\rangle, \quad i, j, k, l = 1, \ldots, 16$$

☞ (R-matrix) Unitarity

$$\boxed{S_{ij}^{mn}(z_1, z_2)\ S_{mn}^{kl}(z_2, z_1) = \delta_i^k \delta_j^l}$$

☞ Crossing symmetry

$$\boxed{S_{ij}^{kl}(z_1, z_2) = S_{\bar{k}i}^{l\bar{j}}(z_2 + \omega_2, z_1)}$$

- Antipode $s : x^{\pm} \to 1/x^{\pm}$ relates $\mathcal{R}_0 \to \mathcal{R}_{\pm 2}$ on the rapidity torus

$$\mathcal{R}_0 \;\to\; \mathcal{R}_{\pm 1} \;\to\; \mathcal{R}_{\pm 2} \quad \equiv \quad \boxed{z \to z \pm \omega_2}$$

- Relativistic S-matrix ($g \to \infty$)
  - Relativistic invariance: $S_{ij}^{kl}(\theta_1, \theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)$
  - ★ Crossing symmetry

    $S_{ij}^{kl}(\theta) = S_{\bar{k}i}^{l\bar{j}}(i\pi - \theta)$ ... or ... $\boxed{S_{ij}^{kl}(\theta_1, \theta_2) = S_{\bar{k}i}^{l\bar{j}}(\theta_2 + i\pi, \theta_1)}$
  - ★ Consistency of the notion of crossing symmetry requires that

$$\boxed{z \to z + \omega_2} \longleftrightarrow \boxed{\theta \to \theta + i\pi} \longrightarrow \boxed{\theta = +\pi u/k}$$

Beisert-Koroteev'08

$$\longmapsto \boxed{\sigma^\gamma(x_1^\pm, x_2^\pm)\,\sigma(x_1^\pm, x_2^\pm) = \frac{x_2^- + \xi}{x_2^+ + \xi} \cdot \frac{x_1^- - x_2^+}{x_1^- - x_2^-} \cdot \frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}}$$

$\sigma^\gamma =$ analytic continuation on $x_1^\pm$ along $\mathcal{R}_0 \to \mathcal{R}_1 \to \mathcal{R}_2$

- Same factorization as in the magnon S-matrix: $\chi(x,y) = -\chi(y,x)$

$$\sigma(x_1^\pm, x_2^\pm) = \exp i\big[\chi(x_1^+, x_2^+) - \chi(x_1^-, x_2^+) - \chi(x_1^+, x_2^-) + \chi(x_1^-, x_2^-)\big]$$

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☞ Solution on $\mathcal{R}_{0,0}$: $|x_1^\pm|, |x_2^\pm| > 1$

Hoare-Hollowood-JLM'11

$$\boxed{\chi(x_1, x_2) = i \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z - x_1} \oint_{|z'|=1} \frac{dz'}{2\pi i} \frac{1}{z' - x_2} \log \frac{\Gamma_{q^2}(1 + iu(z) - iu(z'))}{\Gamma_{q^2}(1 - iu(z) + iu(z'))}}$$

- q-gamma function $\Gamma_{q^2}(1+x) = \frac{1-q^{2x}}{1-q^2}\Gamma_{q^2}(x)$

★ No singularities in $\mathcal{R}_{0,0}$... but branched in $\mathcal{R}_{1,0}$

A puzzle...

☞ Two theories with the same S-matrix but different reality conditions

- Magnon: $\boxed{(x^{\pm})^* = x^{\mp}}$ $\quad U = e^{ip/2g}, \quad V = e^{iE/2g}$
- Mirror: $\boxed{(x^{\pm})^* = 1/x^{\mp}}$ $\quad U = e^{E/2g}, \quad V = e^{p/2g}$

Arutyunov-Frolov'07

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Arutyunov-Frolov'07

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  - Both theories have the same relativistic limit
  - Different spectrum of bound states

- In the $q \longrightarrow 1$ limit, the magnon theory has a bound state pole at

  $\boxed{u_1 - u_2 = -i}$ corresponding to $\boxed{\langle 1, 0 \rangle^{\times 2}}$

  and the mirror theory has a bound state pole at

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- Consistency of crossing requires $\theta = \pi u/k$

  $\xrightarrow{??}$ How to match the bound state poles in the string ($q \to 1$) and relativistic ($g \to \infty$) limits? ☹

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  Hoare-Hollowood-JLM'12

  ★ in the interpolating (non-relativistic) theory, the nature of the bound states depends on their momentum!!

Bound states of the interpolating S-matrix

Hoare-Hollowood-JLM'11
de Leeuw-Matsumoto-Regelskis'11

☞ The interpolating theory is expected to have bound states in the atypical (short) representations

$\boxed{\langle a-1,0\rangle^{\times 2}}$ or $\boxed{\langle 0,a-1\rangle^{\times 2}}$ $a=1,2,\ldots$ of $U_q(\mathfrak{psu}(2|2)^{\oplus 2}\ltimes\mathbb{R}^3)$

- Dispersion relation and central charges

$$q^{-a}\left(x^+ + \frac{1}{x^+}\right) - q^a\left(x^- + \frac{1}{x^-}\right) = (q^a - q^{-a})\left(\xi + \frac{1}{\xi}\right)$$

$$U^2 = q^{-a}\,\frac{x^+ + \xi}{x^- + \xi}, \qquad V^2 = q^{-a}\,\frac{\xi x^+ + 1}{\xi x^- + 1}$$

- Also solved in terms of the map $x(u)$

$$x + \frac{1}{x} + \xi + \frac{1}{\xi} = \frac{1-\xi^2}{\xi}\,q^{-2iu} \;\Rightarrow\; x^\pm = x\left(u \pm \frac{ia}{2}\right)$$

Bound state poles

☞ The S-matrix has two simple poles

- $\mathfrak{su}(2)$ sector: $|\phi^a\phi^a, z_1\rangle \otimes |\phi^a\phi^a, z_2\rangle \longrightarrow |\phi^a\phi^a, z_2\rangle \otimes |\phi^a\phi^a, z_1\rangle$

$$S_{\mathfrak{su}(2)}(z_1, z_2) = \frac{1}{\sigma(z_1, z_2)^2} \cdot \frac{x_1^+ x_2^-}{x_1^- x_2^+} \cdot \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \cdot \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

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- $\mathfrak{sl}(2)$ sector: $|\psi^\alpha\psi^\alpha, z_1\rangle \otimes |\psi^\alpha\psi^\alpha, z_2\rangle \longrightarrow |\psi^\alpha\psi^\alpha, z_2\rangle \otimes |\psi^\alpha\psi^\alpha, z_1\rangle$

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Bound state poles

☞ The S-matrix has two simple poles

- $\mathfrak{su}(2)$ sector: $|\phi^a\phi^a, z_1\rangle \otimes |\phi^a\phi^a, z_2\rangle \longrightarrow |\phi^a\phi^a, z_2\rangle \otimes |\phi^a\phi^a, z_1\rangle$

$$S_{\mathfrak{su}(2)}(z_1, z_2) = \frac{1}{\sigma(z_1, z_2)^2} \cdot \frac{x_1^+ x_2^-}{x_1^- x_2^+} \cdot \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \cdot \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

$\rightsquigarrow$ bound state in $\langle 1, 0\rangle^{\times 2}$

- $\mathfrak{sl}(2)$ sector: $|\psi^\alpha\psi^\alpha, z_1\rangle \otimes |\psi^\alpha\psi^\alpha, z_2\rangle \longrightarrow |\psi^\alpha\psi^\alpha, z_2\rangle \otimes |\psi^\alpha\psi^\alpha, z_1\rangle$

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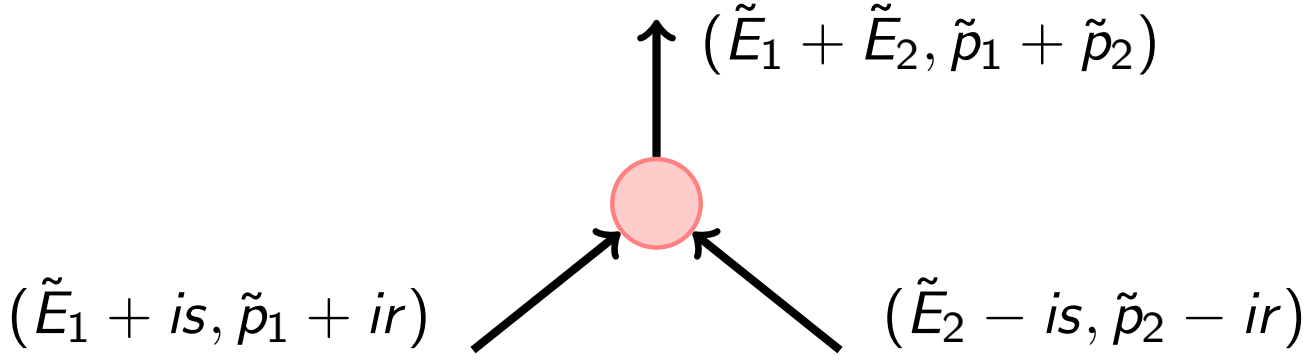
$\rightsquigarrow$ bound state in $\langle 0, 1\rangle^{\times 2}$

☹ No analogue of the physical sheet!!

Bound state equations



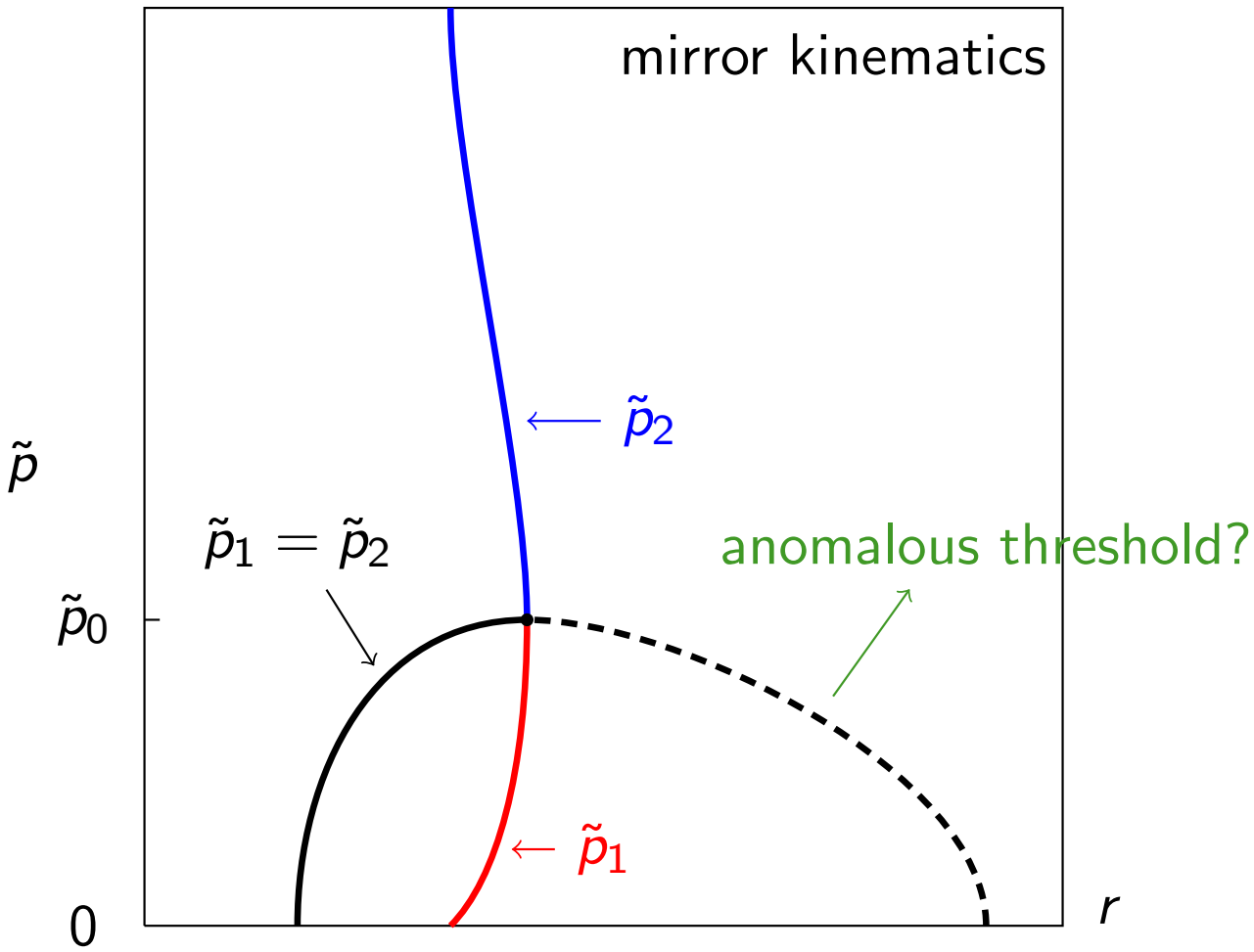
☞ Kinematical conditions

- $\boxed{U_{bs} = U_1 U_2}$ $\boxed{V_{bs} = V_1 V_2}$ $\leftrightarrow$ $E_{bs} = E_1 + E_2, \quad p_{bs} = p_1 + p_2$
- $p_1 = \tilde{p}_1 + ir, \quad p_2 = \tilde{p}_2 - ir$ such that $\boxed{p_{bs} \in \mathbb{R}}$ $\boxed{E_{bs} \in \mathbb{R} > 0}$
  - Relativistic case $\tilde{p}_1 = \tilde{p}_2$ ★
- $\boxed{r > 0}$ Wave function $\Psi(x_1, x_2) = e^{ip_1 x_1 + ip_2 x_2} + S(p_1, p_2) e^{ip_1 x_2 + ip_2 x_1}$

  Reflected wave $|e^{ip_1 x_2 + ip_2 x_1}| = e^{-r(x_2 - x_1)} \to 0$ for $x_1 \ll x_2$

Bound states in the mirror theory

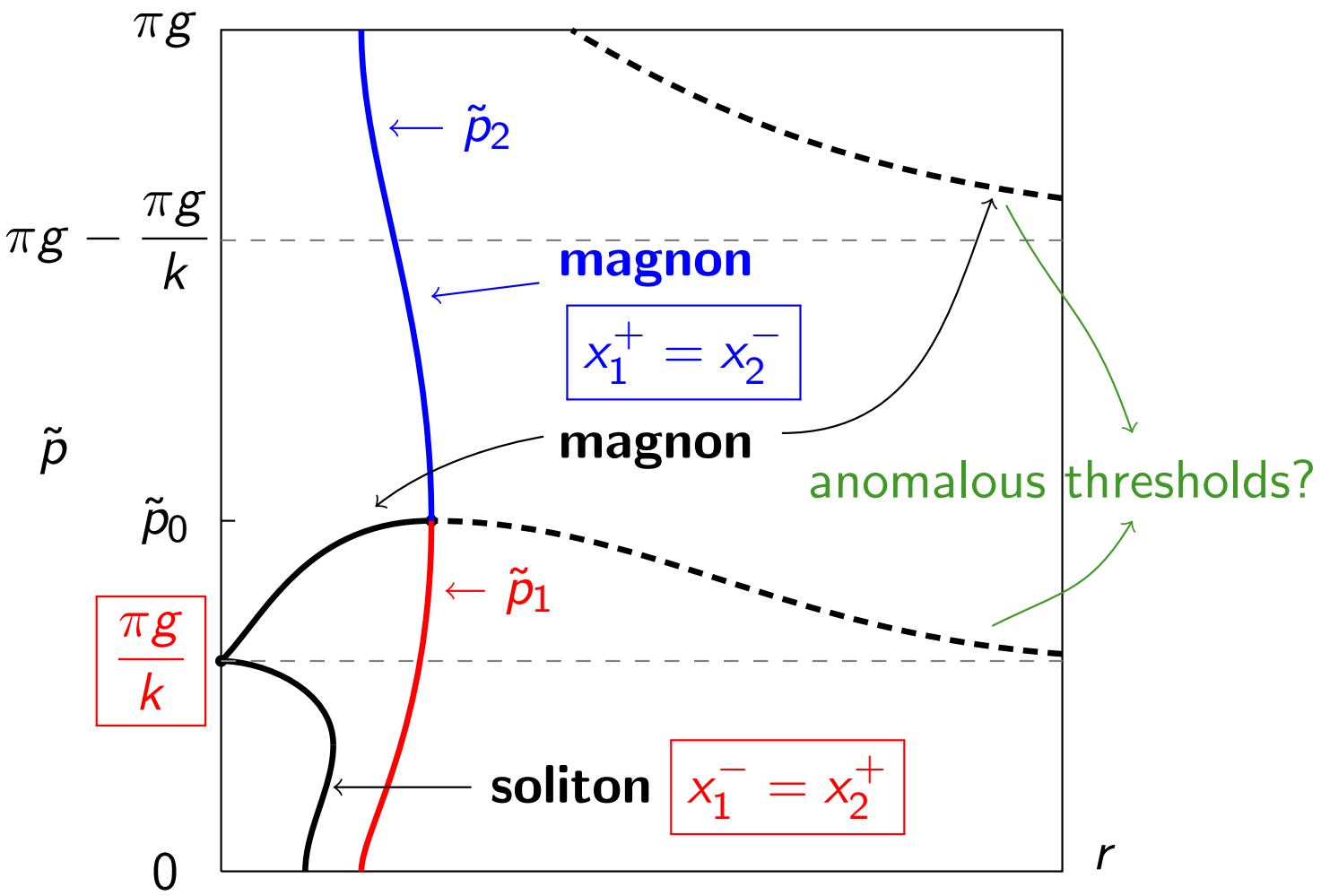
- No solutions for $x_1^+ = x_2^-$
- Two branches for $x_1^- = x_2^+ \implies$ bound states in $\boxed{\langle 0, 1\rangle^{\times 2}}$



Bound states in the magnon theory

- One branch with $x_1^- = x_2^+$ and three branches with $x_1^+ = x_2^-$

★ Bound states in both $\langle 0,1\rangle^{\times 2}$ and $\langle 1,0\rangle^{\times 2}$



- $|p_{bs}| < \frac{2\pi g}{k} \longrightarrow \langle 0, 1\rangle^{\times 2}$

  $\longrightarrow$ Soliton branch (disappears in the $k \to \infty$ limit)

- $|p_{bs}| > \frac{2\pi g}{k} \longrightarrow \langle 1, 0\rangle^{\times 2}$

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$\Rightarrow$ the bound state is marginally unstable for decay into two fundamental particles

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☞ **In relativistic theories, boosts do not change the intrinsic properties of states!!**

Solving the puzzle...

- Bound states in the (string) $k \to \infty$ limit:

  Magnon theory $\longrightarrow$ $\langle a-1, 0\rangle^{\times 2}$

  Mirror theory $\longrightarrow$ $\langle 0, a-1\rangle^{\times 2}$

- Bound states in the (relativistic) $g \to \infty$ limit:

  Magnon and mirror theories $\longrightarrow$ $\langle 0, a-1\rangle^{\times 2}$

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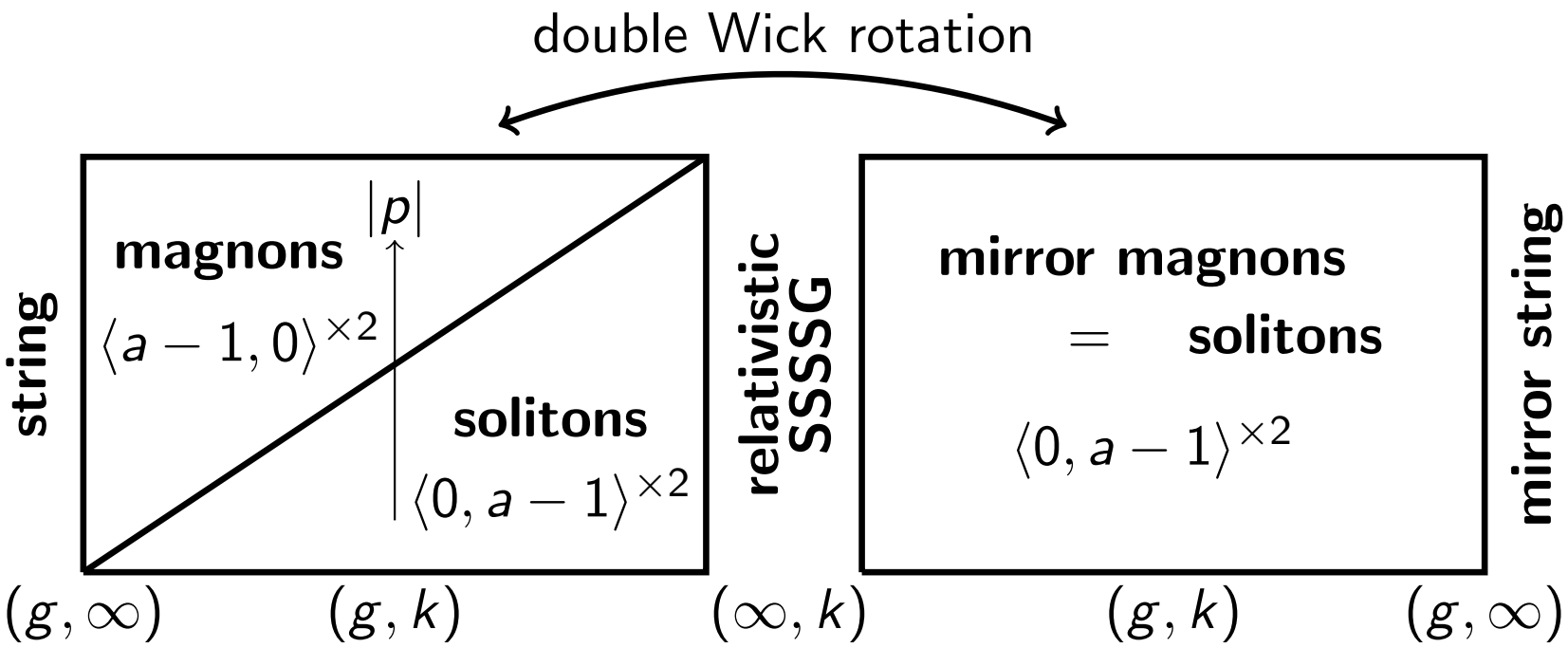

Conclusions

- q-deformed $AdS_5 \times S^5$ superstring S-matrix and its mirror have been constructed

  ☞ Explicit quantum connection between string and PR theories

  ☠ The nature of bound states depends on their momentum

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- Underlying infinite dimensional algebra well understood

  [→ M. de Leeuw's talk]

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★ Applications in the context of the q-deformed Hubbard model