

# Integrable Spin Systems and Representation Theory

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# Introduction

Heisenberg spin  $\frac{1}{2}$  chain (XXX-model)

$$H_{\text{XXX}} = \sum_{m=1}^N (\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + (\sigma_m^z \sigma_{m+1}^z - 1)) = 2 \sum_{m=1}^N (\mathcal{P}_{mm+1} - I_{mm+1}). \quad (1)$$

$$\mathcal{H} = \bigotimes_{m=1}^N \mathbb{C}^2. \quad H_{\text{XXX}} \in \mathbb{C}[\mathcal{S}_N]$$

XXZ-model:

$$H_{\text{XXZ}} = \sum_{k=1}^{N-1} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \cosh \eta (\sigma_k^z \sigma_{k+1}^z - 1)) + \sinh \eta (\sigma_1^z - \sigma_N^z). \quad (2)$$

# Introduction

Heisenberg (XXX)-spin 1/2 chain (1928) ( $su(2)$ -invariant),

Anisotropic XXZ-spin 1/2 chain (1958) [ $su_q(2)$ -invariant, 1990],

QISM:  $\{\sigma_m^\alpha\} \rightarrow \{T_{ij}(\lambda)\}$  (Faddeev-Sklyanin-Takhtajan)

Yang-Baxter equation

$R$ -matrix,  $L$ -operator, boundary  $K$ -matrix

$$H_{XXZ} \in \mathcal{H}_N(q)$$

Algebraic treatment results in

Integrable higher spin  $s = 1, 3/2, 2, \dots$  chains (1980)

$$H_{XXX}^s = \sum_m p_{2s}(x_m), \quad x_m = (S_m^\alpha, S_{m+1}^\alpha)$$

There are three algebras entering the scheme.

# Hecke and Temperley-Lieb Algebras, $\mathcal{B}_N$ , $\mathcal{H}_N(q)$ , $TL_N(q)$ , $BMW(q, \nu)$

Both Hecke algebra  $\mathcal{H}_N(q)$  and TL algebra  $TL_N(q)$  are quotients of the group algebra of the braid group  $\mathcal{B}_N$  generated by  $(N - 1)$  generators  $\check{R}_j$ ,  $j = 1, 2, \dots, N - 1$ , their inverses  $\check{R}_j^{-1}$  and the relations:

$$\check{R}_j \check{R}_k \check{R}_j = \check{R}_k \check{R}_j \check{R}_k, \text{ for } |j-k| = 1 \quad \text{and} \quad \check{R}_j \check{R}_k = \check{R}_k \check{R}_j, \text{ for } |j-k| > 1. \quad (2.1)$$

$$(\check{R}_j - q) (\check{R}_j + 1/q) = 0. \quad (2.2)$$

$$\check{R}_j = q\mathbb{I} + X_j, \quad X_j^2 = -\left(q + \frac{1}{q}\right) X_j. \quad (2.3a)$$

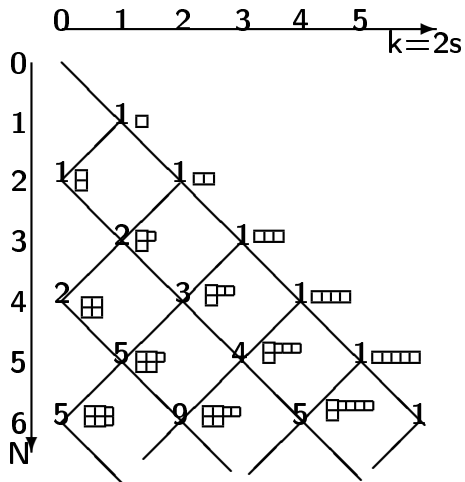
$$X_j X_k X_j - X_j = X_k X_j X_k - X_k, \quad |j - k| = 1. \quad (2.3b)$$

Finally the TL algebra  $TL_N(q)$  is obtained as the quotient algebra of the Hecke algebra  $H_N(q)$  by the set of equations requiring that each side of (2.3b) be zero. To sum up,  $TL_N(q)$  is defined by the generators  $X_j$ ,  $j = 1, 2, \dots, N - 1$  and their relations ( $A_3 = 0$ ):

$$\begin{aligned} X_j^2 &= -\nu(q)X_j, \\ X_j X_k X_j &= X_j, \quad |j - k| = 1, \\ X_j X_k &= X_k X_j, \quad |j - k| > 1 \end{aligned} \quad (2.4)$$

with  $\nu(q) = q + 1/q$ . The dimension of the Hecke algebra,  $N!$ , is the same as the dimension of the symmetric group, whereas the dimension of  $TL_N(q)$  is equal to the Catalan number  $C_N = (2N)!/N!(N+1)!$ . Implementation of the TL constraint thus considerably reduces the dimension of the algebra.

# Schur - Weyl duality $sl(2)$ -case



# Tensor category of $sl(2)$ finite dim. reps

$$sl(2), \quad \{S^z, X^+, X^-\}, \quad s = 0, \frac{1}{2}, 1, \dots;$$

$$\text{IrReps } V_{2s} \quad \dim V_k = k + 1 \quad 2s = k \in \mathbb{Z}_{\geq 0}$$

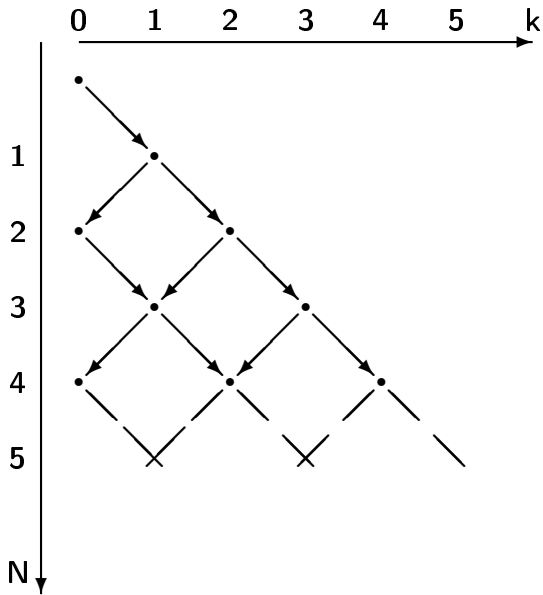
$$V_1 \otimes V_1 = V_0 \oplus V_2;$$

$$V_1 \otimes V_k = V_{k-1} \oplus V_{k+1};$$

$$V_1^{\otimes N} = \bigoplus_{k=0,1}^N V_k \otimes \mathbb{C}^{\nu_k};$$

$$V_l \otimes V_k = \bigoplus_{|l-k|}^{l+k} V_m$$

Multiplicity free "C-G"-decomposition.



This "ring of representations" correspond to some quantum algebra  $\mathcal{U}_q(n)$  if one starts from  $V_0 \simeq \mathbb{C}^1$ ,  $V_1 \simeq \mathbb{C}^n$  (e.g.  $\dim V_2 = n^2 - 1$ )

$$np_k = p_{k+1} + p_{k-1}, \quad p_{-1} = 0, \quad p_0 = 1 \quad \dim V_k = p_k(n).$$

$p_k(x)$  — Chebyshev polynomial of the 2-nd kind.

$V_k$  as corepresentations of dual Hopf algebra and FRT-formalism

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}$$

Girardeau  $\geq 1990$ ; Dubois-Viollet, Bichon, Etingof, Ostrik,...

$$\check{R}_{12} L_1 L_2 = L_1 L_2 \check{R}_{12} \rightarrow \mathcal{U}_q(n).$$

## Ring of irreps: $sl(3)$ case

Highest weight finite dimensional irreducible representations of  $sl(3)$  are parametrized by fundamental weights  $m\omega_1 + n\omega_2$ ,  $V_{mn}$ . According to the Clebsch-Gordan decomposition ( $A_4 = 0$ )

$$V_{10} \otimes V_{mn} = V_{m+1n} \oplus V_{m-1n+1} \oplus V_{mn-1}$$

$$V_{01} \otimes V_{mn} = V_{mn+1} \oplus V_{m+1n-1} \oplus V_{mn-1}$$

$$xp_{mn} = p_{m+1n} + p_{m-1n+1} + p_{mn-1}$$

$$yp_{mn} = p_{mn+1} + p_{m+1n-1} + p_{mn-1}$$

$$sl(3) : x = y = 3, \quad p_{mn} = \frac{1}{2}(m+1)(n+1)(m+n+2), \quad p_{00} = 1.$$

# Recurrent relations for generalized Chebyshev polynomials

$$xp_{m,0}(x, y) = p_{m+1,0}(x, y) + yp_{m-1,0}(x, y) - p_{m-2,0}(x, y).$$

$$p_{m,0}(x, y) = x^m - (m-1)x^{m-2}y + \frac{(m-3)(m-2)}{2}y^2 - b_mx^{m-6}y^3 + \dots$$

The generating function for polynomials  $p_{m,0}(x, y)$  looks as

$$\varphi(t; x, y) = \frac{1}{1 - tx + yt^2 - t^3}.$$

For more general polynomials we have the following recurrent relations

$$xp_{m,n}(x, y) - yp_{m-1,n}(x, y) = p_{m+1,n}(x, y) - p_{m-2,n}(x, y).$$

The matrix realization on  $\mathcal{H}$  of the idempotent generator  $X_j$  now reads in terms of  $b$ :

$$X_j = \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{j-1} \otimes \left( \sum_{\substack{c, d, c', d' \\ \in \{1 \dots n\}}} b_{cd} \bar{b}_{c'd'} E_{cc'} \otimes E_{dd'} \right) \otimes \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{N-j-1} \quad (2.5)$$

where we have now denoted by  $\mathbb{I}$  the identity matrix in  $End(\mathbb{C}^n)$  and we have used the canonical basis of  $n \times n$  matrices,  $E_{cc'}$  denoting the  $n \times n$  matrix with entries  $(E_{cc'})_{xx'} = \delta_{cx} \delta_{c'x'}$ ,  $\check{R} = q\mathbb{I} + X$ .

Direct computation shows that the set of relations (2.4) are satisfied and they fix the value of the parameter  $q$  up to a duality  $q \rightarrow 1/q$ :

$$-\nu(q) = \text{tr}^t b \bar{b} = - \left( q + \frac{1}{q} \right). \quad (2.6)$$

# Yang-Baxterization process

$$\check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23}. \quad (2.7)$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \quad (2.8)$$

$$\check{R}_j(u) = u\check{R}_j - \frac{1}{u}\check{R}_j^{-1} = \left(u - \frac{1}{u}\right)\check{R}_j + \frac{\omega(q)}{u}\mathbb{I}; \quad \omega(q) = q - \frac{1}{q}. \quad (2.9)$$

$$\check{R}_j(u)\check{R}_k(uw)\check{R}_j(w) = \check{R}_k(w)\check{R}_j(uw)\check{R}_k(u), \quad \text{for } |j - k| = 1. \quad (2.10)$$

$$R_{12}(u)R_{13}(uw)R_{23}(w) = R_{23}(w)R_{13}(uw)R_{12}(u). \quad (2.11)$$

# Transfer matrix and spectrum

$$\check{R}(u) \rightarrow R(u) = L_{a,j}(u) \rightarrow T_N(u),$$
$$t(u) = \text{Tr} T(u) = \omega(uq)^N U + \sum_j Y_j + \omega(u)^N U^{-1},$$

$$\omega(u) = u - u^{-1}, \quad \omega \in \mathbb{C}^n : Y_j \omega \otimes \omega = 0, \quad \Omega = \bigotimes_1^N \omega$$

$$t(u)\Omega = \Lambda_0(u)\Omega, \quad \Lambda_0(u) = \omega(uq)^N + \omega(u)^N.$$

Functional relations of  $T^{(k)}(\omega)$  and analytic Bethe ansatz

$$\Lambda(u; \{v_j\}_1^M) = \omega(uq)^N \prod_j^M \frac{\omega(u/qv_j)}{\omega(u/v_j)} + \omega(u)^N \prod_j^M \frac{\omega(uq/v_j)}{\omega(u/v_j)}.$$

# Classification of the solutions of the constant RE

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21} \quad (3.1)$$

$$R = \mathcal{P}\check{R}, \quad \check{R}_{12} K_1 \check{R}_{12} K_1 = K_1 \check{R}_{12} K_1 \check{R}_{12}. \quad (3.2)$$

$$\check{R} = q\mathbb{I} + X, \quad qX_{12}K_1^2 + X_{12}K_1X_{12}K_1 = qK_1^2X_{12} + K_1X_{12}K_1X_{12}. \quad (3.3)$$

This equation reads in term of the  $b$  matrix:

$$qb \otimes ({}^t K^2 \bar{b}) + \text{tr}({}^t b^t K \bar{b}) b \otimes ({}^t K \bar{b}) = q({}^t K^2 b) \otimes \bar{b} + \text{tr}({}^t b^t K \bar{b}) ({}^t K b) \otimes \bar{b}. \quad (3.4)$$

Since matrices  $b$  and  $\bar{b}$  are invertible

$$\mathbb{I} \otimes (q K^2 + \text{tr}({}^t b^t K \bar{b}) K) = (q K^2 + \text{tr}({}^t b^t K \bar{b}) K) \otimes \mathbb{I}. \quad (3.5)$$

This establishes that  $q K^2 + \text{tr}({}^t \bar{b} K b) K$  is proportional to identity.

$$K^2 + \frac{1}{q} \text{tr}({}^t \bar{b} K b) K = k_2 \mathbb{I}, \quad k_2 = \frac{1}{q \text{tr}({}^t \bar{b} b)} ((\text{tr}({}^t \bar{b} K b))^2 + q \text{tr}({}^t \bar{b} K^2 b)). \quad (3.6)$$

the complete resolution of the reflection equation for these constant TL  $R$ -matrices will be realized in two steps:

1. Parametrize all matrices  $K$  with a minimal polynomial of degree 2 (or less).
2. Fix the value of the coefficient of the linear term to its expression in (3.6).

Step 1 is separated into three obvious subcases:

1a: Minimal polynomial of degree 1. The matrix  $K$  is then proportional to the Identity and automatically solves the reflection equation without further conditions.

1b: Minimal polynomial of degree 2 with two distinct roots. The matrix  $K$  is then diagonalizable with the same two zeroes as eigenvalues.

1c: Minimal polynomial of degree 2 with a double root. The matrix  $K$  is then only trigonalizable (i.e. is written with Jordanian cells)

$$\sqrt{2}X = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$$

$$\check{R}(q) = qP_+ - \frac{1}{q}P_-, \quad \text{rank}P_{\pm} = 2, \quad q = \frac{i-1}{\sqrt{2}}$$

Not "skew invertible",  $\det(\mathcal{P}\check{R})^{t_1} = 0$

$$\sqrt{3}X = \left( \begin{array}{cc|cc|cc} 1 & & & m & & 1 \\ & 1 & & & & \\ & & 1 & m & & \\ \hline & & & 1 & 1 & \\ m^2 & & & & & 1 \\ & & & & & \\ \hline & & & & 1 & \\ m^2 & & & 1 & & m^2 \\ \hline & & 1 & & 1 & \\ & & & & & \\ 1 & & & & m & 1 \\ & & & & & \\ & m^2 & 1 & & & 1 \end{array} \right), \quad m^3 = 1$$

$$\check{R}(q) = qP_+ - \frac{1}{q}P_-, \quad \text{rank}P_- = 3, \quad q = \frac{i - \sqrt{3}}{2}$$

# R-matrix of XXZ spin-1 chain

R-matrix of the XXZ-chain of spin one can be expressed as follows

$$R(\lambda, \eta) = \left( \begin{array}{c|c|c} a_1 & & \\ \hline & a_2 & b_1 \\ & a_3 & b_2 \\ \hline c_1 & a_2 & b_3 \\ & c_2 & b_2 \\ & & a_2 & b_1 \\ \hline & c_3 & c_2 & a_3 \\ & & c_1 & a_2 \\ & & & a_1 \end{array} \right), \quad (3)$$

where the functions are expressed in trigonometric functions e.g.

$$a_1 = \sinh(\lambda + \eta) \sinh(\lambda + 2\eta), \quad b_2 = e^\lambda \sinh \lambda \sinh 2\eta,$$

The R-matrix (1) satisfies the Yang-Baxter equation in the space  $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

$$R_{12}(\lambda)R_{13}(\lambda + \mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda + \mu)R_{23}(\lambda), \quad (4)$$

The R-matrix has a few important properties: regularity, unitarity, PT-symmetry and crossing symmetry. The regularity condition at  $\lambda = 0$  reads

$$R(0, \eta) = \sinh(\eta) \sinh(2\eta) \mathcal{P}, \quad (5)$$

where  $\mathcal{P}$  is the permutation matrix of  $\mathbb{C}^3 \otimes \mathbb{C}^3$ . The unitarity relation is

$$R_{12}(\lambda)R_{21}(-\lambda) = \rho(\lambda)\mathbf{1}, \quad (6)$$

here  $R_{21}(\lambda) = \mathcal{P}R_{12}(\lambda)\mathcal{P}$  and  $\rho$  is the following function

$$\rho(\lambda) = \sinh(\lambda + \eta) \sinh(\lambda + 2\eta) \sinh(\lambda - \eta) \sinh(\lambda - 2\eta)$$

The so-called PT-symmetry states

$$R_{12}^t(\lambda) = R_{21}(\lambda). \quad (7)$$

Finally, it has the following crossing symmetry property

$$R(\lambda) = (Q \otimes \mathbf{1}) R^{t_2}(-\lambda - \eta) (Q \otimes \mathbf{1}), \quad (8)$$

where  $t_2$  denotes the transpose in the second space and the matrix  $Q$  is given by

$$Q = \begin{pmatrix} 0 & 0 & -e^{-\eta} \\ 0 & 1 & 0 \\ -e^{\eta} & 0 & 0 \end{pmatrix}. \quad (9)$$

The R-matrix  $R$  in the braid group form  $\check{R}(\lambda, \eta) = \mathcal{P}R(\lambda, \eta)$  admits the spectral decomposition

$$\check{R}(\lambda, \eta) = \sum \rho_k(\lambda, \eta) P_k(\eta), \quad k = 1, 3, 5; \quad \text{rank} P_k(\eta) = k.$$

The R-matrix can also be expressed in the following form, useful for the asymptotics  $\check{R}(\lambda, \eta) = \frac{e^\eta}{4} (e^{2\lambda} - 1) \check{R}(\eta) + (\sinh \eta \sinh 2\eta) \mathbf{1} + \frac{e^{-\eta}}{4} (e^{-2\lambda} - 1) \check{R}^{-1}(\eta)$ . A relevant observation is that the constant R-matrix is a solution of the Yang-Baxter equation in the braid group form

$$\check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23}. \quad (10)$$

It has the spectral decomposition ( $q = e^{2\eta}$ )

$$\check{R}(\eta) = qP_5(\eta) - \frac{1}{q}P_3(\eta) + \frac{1}{q^2}P_1(\eta), \quad (11)$$

and  $\check{R}(\eta)$  satisfies the cubic equation

$$(\check{R}(\eta) - q\mathbf{1}) \left( \check{R}(\eta) + \frac{1}{q}\mathbf{1} \right) \left( \check{R}(\eta) - \frac{1}{q^2}\mathbf{1} \right) = 0. \quad (12)$$

# Spectral parameter dependent $K$ matrices.

## Yang-baxterization for the associated $K$ -matrices

the affine Hecke algebra  $\mathcal{H}_N(q)$ . It has one more generator  $K$  with relations:

$$\check{R}_1 K \check{R}_1 K = K \check{R}_1 K \check{R}_1 \quad K \check{R}_j = \check{R}_j K, j > 1. \quad (4.1)$$

As in the Hecke case will define a quotient of  $\hat{H}_N(q)$ . There exists then consistent realizations of the Yang-Baxterized  $K$ -matrix by Laurent polynomials  $K(u)$  in  $u, u^{-1}$ , depending on the coefficients of  $p_n$  and  $K^m$ ,  $m = 0, 1, \dots, n - 1$ . They are solutions to the algebraic reflection equation:

$$\check{R}_1(u/w) K(u) \check{R}_1(uw) K(w) = K(w) \check{R}_1(uw) K(u) \check{R}_1(u/w) \quad (4.2)$$

where  $\check{R}_1(u/w)$  is the Yang-Baxterized  $\check{R}$  matrix.

Suitable matrix representations of both  $R$  and  $K$  are considered, respectively in  $End(\mathbb{C}^n \otimes \mathbb{C}^n)$  and  $End(\mathbb{C}^n)$  becomes the well-known Sklyanin reflection equation

$$\check{R}_{12}(u/w)K_1(u)\check{R}_{12}(uw)K_1(w) = K_1(w)\check{R}_{12}(uw)K_1(u)\check{R}_{12}(u/w). \quad (4.3)$$

$K(u)$  is given by the expression:

$$K(u) = u^2 K - \frac{1}{u^2} K^{-1} + c\mathbb{I} \quad (4.4)$$

with an arbitrary central element  $c$ . After a suitable normalization of  $K$ , one gets the regularity property of  $K(u)$ :  $K(u)|_{(u=1)} = \mathbb{I}$ . A boundary interaction on the left and right boundary sites described by matrices  $K^-(u)$  and  $K^+(u)$  respectively.



# Birman-Wenzl-Murakami algebra $W_N(q, \nu)$

The defining relations of the BMW algebra  $W_N(q, \nu)$ , for the generators  $1, \sigma_i, \sigma_i^{-1}$  and  $e_i, i = 1, \dots, N - 1$ , are recalled for convenience,

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & \sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ for } |i - j| > 1, \\ e_i \sigma_i &= \sigma_i e_i = \nu e_i, & e_i \sigma_{i-1}^{\pm 1} e_i &= \nu^{\mp 1} e_i, \\ \sigma_i - \sigma_i^{-1} &= \omega(q)(1 - e_i),\end{aligned}$$

where  $\omega(q) = q - 1/q$ . It can be shown that the dimension of the Birman-Wenzl-Murakami algebra  $W_N(q, \nu)$  is  $(2N - 1)!!$ .

Many useful relations follow from the definition above. f.ex.

$$e_i^2 = \mu e_i, \text{ with } \mu = \frac{\omega - \nu + 1/\nu}{\omega} = \frac{(q - \nu)(\nu + 1/q)}{\nu \omega}.$$

Another important consequence of the relations is a cubic one

$$(\sigma_i - q)(\sigma_i + q^{-1})(\sigma_i - \nu) = 0.$$

There is the natural inclusion of  $W_M(q, \nu) \subset W_N(q, \nu)$ ,  $M < N$ .  
 Namely, the first  $3(M - 1)$  generators  $\{\sigma_i^{\pm 1}, e_i; i = 1, 2, \dots, M - 1\}$   
 of  $W_N(q, \nu)$  define the algebra  $W_M(q, \nu)$ .

The Yang-Baxterization procedure yields two spectral parameter  
 dependent algebraic elements

$$\sigma_i^{(\pm)}(u) = \frac{1}{\omega} (u^{-1} \sigma_i - u \sigma_i^{-1}) + \frac{\nu \pm q^{\pm 1}}{u\nu \pm q^{\pm 1} u^{-1}} e_i. \quad (14)$$

These elements satisfy the Yang-Baxter equation in the braid group  
 form

$$\sigma_i^{(\pm)}(u) \sigma_{i+1}^{(\pm)}(uv) \sigma_i^{(\pm)}(v) = \sigma_{i+1}^{(\pm)}(v) \sigma_i^{(\pm)}(uv) \sigma_{i+1}^{(\pm)}(u). \quad (15)$$

Their unitarity relation is

$$\sigma_i^{(\pm)}(u) \sigma_i^{(\pm)}(u^{-1}) = (1 - \omega^{-2}(u - u^{-1})^2). \quad (16)$$

The irreducible representations of the BMW algebra  $W_N(q, \nu)$  are more complicated than the irreducible representations of the symmetric group  $\mathfrak{S}_N$  or the Hecke algebra  $\mathcal{H}_N(q)$ , although they can be parameterized by the Young diagrams. The simplest, one-dimensional irreducible representations of  $W_N(q, \nu)$  are defined by the symmetrizer and antisymmetrizer, respectively. The symmetrizer of the  $W_N(q, \nu)$  is given by

$$\mathcal{S}_N = \frac{1}{[N]_q!} \sigma_1^{(-)}(q^{-1}) \sigma_2^{(-)}(q^{-2}) \cdots \sigma_{N-1}^{(-)}(q^{-(N-1)}) \mathcal{S}_{N-1}, \quad (17)$$

with  $\mathcal{S}_1 = 1$  and

$$\mathcal{S}_2 = \frac{1}{[2]_q} \sigma_1^{(-)}(q^{-1}). \quad (18)$$

Similarly, the antisymmetrizer of the  $W_N(q, \nu)$  is given by

$$\mathcal{A}_N = \frac{1}{[N]_q!} \sigma_1^{(+)}(q) \sigma_2^{(+)}(q^2) \cdots \sigma_{N-1}^{(+)}(q^{N-1}) \mathcal{A}_{N-1}, \quad (19)$$

with  $\mathcal{A}_1 = 1$  and

$$\mathcal{A}_2 = \frac{1}{[2]_q} \sigma_1^{(+)}(q). \quad (20)$$

The elements  $\mathcal{A}_n$ ,  $n = 1, \dots, N$  are idempotents and the antisymmetrizer  $\mathcal{A}_N$  is also central in  $W_N(q, \nu)$ .

It is straightforward to see that

$$\mathcal{A}_3 \simeq \sigma_1^{(+)}(q)\sigma_2^{(+)}(q^2)\sigma_1^{(+)}(q) = \sigma_2^{(+)}(q)\sigma_1^{(+)}(q^2)\sigma_2^{(+)}(q). \quad (21)$$

In this realisation

$$\sigma_1 = \check{R}(\eta) = qP_5 - q^{-1}P_3 + \nu P_1, \quad \nu = \frac{1}{q^2},$$

$$e_1 = \mu P_1 = (q + 1 + q^{-1})P_1,$$

$$\sigma_1^{(+)}(q) = [2]_q P_3,$$

$$\sigma_1^{\pm 1} P_3 = -q^{\mp 1} P_3,$$

$$e_1 P_3 = 0.$$

The BMW algebra  $W_N(q, q^{-2})$  will be used to describe the multiplet structure of the spectra of some open quantum spin chains.

# Open Spin Chain

According to the QISM the R-matrix  $R(u, q)$  can be used to construct an auxiliary L-operator for an integrable spin system, identifying the two spaces of  $R(u, q) \in \text{End}(V \otimes V)$  as auxiliary and quantum space, respectively:

$$L_{0j}(u) = R_{0j}(u, q). \quad (22)$$

the monodromy matrix of a spin chain with N sites is the product of L-matrices in  $\text{End}(V_0)$  whose entries are in  $\text{End}(V_j)$

$$T(u) = L_{0N}(u)L_{0N-1}(u) \cdots L_{01}(u),$$

while the entries of the monodromy matrix  $T_{ab}(u)$  are operators on the whole space of states  $\mathcal{H} = \otimes_{j=1}^N V_j$  (in the case under consideration  $V_j = \mathbb{C}^3$ ).

As a consequence of the Yang-Baxter equation for the R-matrix and (22) one has

$$R_{00'} \left( \frac{u}{w} \right) L_{0j}(u) L_{0'j}(w) = L_{0'j}(w) L_{0j}(u) R_{00'} \left( \frac{u}{w} \right) \quad (23)$$

and

$$R_{12} \left( \frac{u}{w} \right) T_1(u) T_2(w) = T_2(w) T_1(u) R_{12} \left( \frac{u}{w} \right), \quad (24)$$

where  $T_1(u) = T(u) \otimes \mathbb{I}$  and  $T_2(u) = \mathbb{I} \otimes T(u)$  are operator valued matrices in the two auxiliary spaces  $V_1 \otimes V_2$ , written as elements of  $\text{End}(V_1 \otimes V_2)$ . The trace of the monodromy matrix  $T(u)$  - the transfer matrix

$$t(u) = \text{tr}_0 T(u), \quad (25)$$

is the generating function of the integrals of motion, including the Hamiltonian, of the spin chain with the periodic boundary condition.

In order to construct integrable spin chains with non-periodic boundary condition one has to use the Sklyanin formalism. The corresponding monodromy matrix  $\mathcal{T}(u)$  consists of the two matrices  $T(u)$  and a reflection matrix  $K^-(u) \in \text{End}(V)$

$$\mathcal{T}(u) = T(u)K^-(u)T^{-1}(u^{-1}).$$

Using the unitarity relation  $R_{12}^{-1}(u^{-1}) = R_{21}(u)$  one gets

$$T^{-1}(u^{-1}) = R_{10}(u)R_{20}(u) \cdots R_{N0}(u).$$

Taking into account the definition  $R_{12}(u, \eta) = \mathcal{P}_{12}R_{21}(u, \eta)\mathcal{P}_{12}$  one can transform the monodromy matrix  $\mathcal{T}(u)$  into the following form

$$\mathcal{T}(u) = \check{R}_{N0}(u)\check{R}_{N-1N}(u) \cdots \check{R}_{12}(u)K_1^-(u)\check{R}_{12}(u)\check{R}_{23}(u) \cdots \check{R}_{N0}(u).$$

The generating function  $\tau(u)$  of the integrals of motion is given by the trace of  $\mathcal{T}(u)$  over the auxiliary space with an extra reflection matrix  $K^+(u)$

$$\tau(u) = \text{tr}_0 (K_0^+(u)\mathcal{T}(u)).$$

The reflection matrices  $K^\pm(u)$  are solutions to the reflection equation with a property  $K^-(1) = \mathbf{1} \in \text{End}(V)$  and  $\tau(1) \simeq \mathbf{1}$ . In particular, the Hamiltonian is given by  $H = \frac{1}{2} \frac{d}{du} \ln \tau(u)|_{u=1}$ ,

$$H = \sum_{i=1}^{N-1} \check{R}'_{i,i+1}(1) + \frac{\text{tr}_0 K_0^+(1) \check{R}'_{N0}(1)}{\text{tr}_0 K_0^+(1)} + \frac{1}{2} \left( \frac{dK_1^-(1)}{du} + \frac{1}{\text{tr}_0 K_0^+(1)} \frac{d \text{tr}_0 K_0^+(1)}{du} \right).$$

The Hamiltonian density  $h_{i,i+1} = \frac{d}{du} \check{R}_{i,i+1}(u)|_{u=1}$  is a function of the generators of  $W_N(q, q^{-2})$  on the space  $\mathcal{H} = \otimes_1^N \mathbb{C}^3$ . The two extra boundary terms are contributions from the two reflection matrices  $K^\pm(u)$  at the sites 1 and  $N$ . In our case we can take the constant K-matrices  $K^-(u) = 1$  and  $K^+(u) = Q^t Q$ , where the matrix  $Q$  is given by (7). It is easy to check that a non-zero contribution at the site  $N$  is proportional to the identity, hence it does not influence the structure of the spectrum. For general K-matrices the solution, by the algebraic Bethe ansatz, was given by Kurak and Lima-Santos. Asymptotic expansion of  $T(u)$  at  $u \rightarrow 0$  (or at  $u \rightarrow \infty$ ) results in some matrices which have no spectral parameter dependence

$$T(u) = u^{-N} L_{0N}^- L_{0,N-1}^- \cdots L_{01}^- + \mathcal{O}(u^{-N+1}).$$

Here the constant L-matrices  $L_{0j}^-$  are upper triangular matrices which coincide with the asymptotic limit  $\lambda \rightarrow +\infty$  of the R-matrices

$$L_{0j}^- = R_{0j}^- = \mathcal{P}_{0j} \check{R}_{0j}.$$

Hence, the Yang-Baxter equation for the constant R-matrix can be written as follows  $R_{i,i+1}^- L_{0,i+1}^- L_{0i}^- = L_{0i}^- L_{0,i+1}^- R_{i,i+1}^-$ . It follows that  $R_{i,i+1}^- = \mathcal{P}_{i,i+1} \check{R}_{i,i+1}$ . So, multiplying the previous equation by the permutation operator  $\mathcal{P}_{i,i+1}$  from the left one gets

$$[\check{R}_{i,i+1}, L_{0,i+1}^- L_{0i}^-] = 0.$$

It is then obvious that  $\rho_W(\sigma_i) = \check{R}_{i,i+1} \equiv \check{R}_i$ ,  $\rho_W(e_i) = \mu(P_1(\eta))_{i,i+1}$  as the representation  $\rho_W$  of the generators of the BMW algebra  $\mathcal{W}_N(q, q^{-2})$  in the space  $\mathcal{H} = \otimes_1^N \mathbb{C}^3$ , commute with the generators  $T_{ab}^-$  of the global (or diagonal) action of the quantum algebra  $\mathcal{U}_q(\mathfrak{o}(3))$  on the space  $\mathcal{H}$

$$[\check{R}_{i,i+1}, T^-] = 0, \quad T^- = L_{0N}^- L_{0,N-1}^- \cdots L_{01}^-.$$

This product of  $L_{0j}^-$  can be represented as the image of a multiple co-product map  $\Delta^N : \mathcal{U}_q(\mathfrak{o}(3)) \rightarrow (\mathcal{U}_q(\mathfrak{o}(3)))^{\otimes N}$  acting on a universal L-matrix  $\mathcal{L}_0^-$  with entries in  $\mathcal{U}_q(\mathfrak{o}(3))$  on the representation space  $\mathcal{H}$

$$T^- = (\text{id} \otimes \rho_W)(\text{id} \otimes \Delta^N) \mathcal{L}_0^-.$$

It is known that in the space  $\mathcal{H}$  as a space of representation of  $\mathcal{U}_q(\mathfrak{o}(3))$  and  $W_N(q, q^{-2})$  these algebras are mutual centralizers. According to the centralizer property this induces the decomposition of the representation space  $\mathcal{H}$  into direct sum of irreducible representations of both algebras, being a generalisation of the Schur-Weyl duality. Similarly to the Hecke algebra case one gets

$$\mathcal{H} = \sum_{s=0}^N V_s \otimes U_s,$$

where  $V_s$  is the  $(2s + 1)$ -dimensional irreducible representation of  $\mathcal{U}_q(\mathfrak{o}(3))$  while  $U_s$  is some irreducible representation of  $W_N(q, q^{-2})$ . The dimension of an irreducible representation of  $W_N(q, q^{-2})$  is equal to the multiplicity  $m$  of the corresponding irreducible representation of centralizer algebra  $\mathcal{U}_q(\mathfrak{o}(3))$ , and vice versa

$$m(V_s) = \dim U_s, \quad m(U_s) = \dim V_s.$$

The decomposition  $\mathcal{H} = \sum_{s=0}^N V_s \otimes U_s$  permits to determine the structure of the multiplets of the Hamiltonian, which is an element of the BMW algebra being a function of the generators of  $W_N(q, q^{-2})$

$$H = \sum_{i=1}^{N-1} h_{i,i+1}, \quad h_{i,i+1} = \frac{d}{d\lambda} \check{R}(\lambda, \eta)|_{\lambda=0} = f(\check{R}_i) \in W_N(q, q^{-2}).$$

According to the QISM, the R-matrices being regular at  $\lambda = 0$  define the local Hamiltonian density for two sites of the corresponding spin chains. For the  $XXZ_1$ -model from one gets

$$\begin{aligned} h_{XXZ} &= \frac{d}{d\lambda} \check{R}(\lambda, \eta)|_{\lambda=0} \simeq q\check{R}(\eta) - \check{R}^{-1}(\eta) \\ &= (q-1) \left( \left( q+1 + \frac{1}{q} \right) (P_5 - P_1) + P_3 \right). \end{aligned}$$

In the  $A_2^{(2)}$ -case from it follows

$$h_A = \frac{d}{d\lambda} \check{R}(\lambda, \eta)|_{\lambda=0} \simeq q\check{R}(\eta) + \frac{1}{q^2} \check{R}^{-1}(\eta) = (q^2 + \frac{1}{q^3})P_5 + (1 + \frac{1}{q})(P_1 - P_3).$$

The Hamiltonian of the open spin chain with  $N$ -sites is then given by

$$H = \sum_{i=1}^{N-1} h_{i,i+1}.$$

As an example let us consider the case of  $N = 3$  sites when the algebra  $W_3(q, 1/q^2)$  is realised on  $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$  and the corresponding Hamiltonians are  $H = h_{12} + h_{23}$ . From the relations of the BMW-algebra it follows

$$H_{XXZ} \mathcal{S}_3 = 2(q + 1 + \frac{1}{q})\mathcal{S}_3, \quad H_{XXZ} \mathcal{A}_3 = 2\mathcal{A}_3$$

and similarly for the  $H_A$

$$H_A \mathcal{S}_3 = 2(q^2 + \frac{1}{q^3})\mathcal{S}_3, \quad H_A \mathcal{A}_3 = -2(1 + \frac{1}{q})\mathcal{A}_3.$$

In the case  $N = 3$  there are four irreducible representations (irreps) of  $W_3$ : two one-dimensional irreps generated by  $\mathcal{S}_3$  and  $\mathcal{A}_3$ , respectively, the three-dimensional irrep  $d_3$  (corresponding to the one-box Young diagram) and the two-dimensional irrep  $d_2$  (corresponding to the three-box Young diagram with two rows). Thus the Hamiltonian being restricted to invariant subspaces can have up to seven distinct eigenvalues. Their multiplicities are obtained from the correspondence between the irreps of  $W_3$  and the irreps of  $\mathcal{U}_q(\mathfrak{o}(3))$ :

$$U(\mathcal{S}_3) \sim V_3, \quad U(\mathcal{A}_3) \sim V_0 \quad U(d_3) \sim V_1 \quad U(d_2) \sim V_2.$$

The degeneracies of corresponding energy values are

$$m(\epsilon(\mathcal{S}_3)) = 7, \quad m(\epsilon(\mathcal{A}_3)) = 1, \quad m(\epsilon_j(d_3)) = 3, \quad m(\epsilon_k(d_2)) = 5,$$

where  $j = 1, 2, 3$ ;  $k = 1, 2$ .

The exact values of the corresponding energy are obtained by direct calculations and are given below. For the XXZ-model of spin 1 the corresponding expressions are

$$\epsilon(\mathcal{S}_3) = 2\left(q + 1 + \frac{1}{q}\right), \quad \epsilon(\mathcal{A}_3) = 2,$$

$$\epsilon_1(d_3) = 1, \quad \epsilon_{2,3}(d_3) = \left(\frac{1}{2} \pm \sqrt{\frac{1}{2} + 2\left(q + 3 + \frac{1}{q}\right)}\right),$$

$$\epsilon_1(d_2) = \left(q + 1 + \frac{1}{q}\right), \quad \epsilon_2(d_2) = \left(q + 3 + \frac{1}{q}\right).$$

In the  $A_2^{(2)}$ -case the corresponding expressions are

$$\epsilon(\mathcal{S}_3) = 2\left(q^2 + \frac{1}{q^3}\right), \quad \epsilon(\mathcal{A}_3) = -2\left(1 + \frac{1}{q}\right),$$

$$\epsilon_1(d_3) = \left(q^2 + \frac{1}{q^3}\right),$$

$$\epsilon_{2,3}(d_3) = \frac{1}{2} \left( \left(q^2 + \frac{1}{q^3}\right) \pm \sqrt{q^4 + 8q^2 - 8q + \frac{34}{q} - \frac{8}{q^3} + \frac{8}{q^4} + \frac{1}{q^6}} \right),$$

$$\epsilon_1(d_2) = \left(1 + \frac{1}{q}\right)\left(q^2 - 1 + \frac{1}{q^2}\right),$$

$$\epsilon_2(d_2) = \left(1 + \frac{1}{q}\right)\left(q^2 - 2q + 1 - \frac{2}{q} + \frac{1}{q^2}\right).$$

# Conclusions

We point out that there is a series of quantum super-algebras  $\mathcal{U}_q(\mathfrak{osp}(1|2n))$  with corresponding R-matrices in the vector representation defining generators of the Birman-Wenzl-Murakami algebra  $W_N(-q, -q^{-2n})$ . From the representation of the BMW algebra given by the R-matrix one can get two spectral parameter dependent R-matrices by the Yang-Baxterization procedure. Each of them yields solutions of the standard and ( $\mathbb{Z}_2$  graded) super-Yang-Baxter equations. This results in a possibility to construct four series of integrable spin chains whose structure of the spectrum is similar to the one considered above.

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