The quantum Ablowitz-Ladik model: completeness of the Bethe ansatz

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Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra. Communications in Mathematical Physics (in press) **arxiv: 1110.6356**

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Outline

1. q-boson (Heisenberg) algebra and quantum Ablowitz-Ladik
2. exactly solvable lattice models
3. TQ-relation and Bethe ansatz equations
4. Combinatorial solution: spherical Hecke algebra and affine straightening rules
5. Summary: from quantum many body systems to 2D TFT

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Quantum Ablowitz-Ladik equation

Classical (defocusing) Ablowitz-Ladik equation: $\psi_j \in \mathbb{C}$, $|\psi_j| < 1$

$$-i\dot{\psi}_j = (1 - |\psi_j|^2)(\psi_{j+1} + \psi_{j-1}) - 2\psi_j, \quad \psi_{j+n} = \psi_j$$

Definition (q-deformed boson algebra)

$\mathcal{H}_q$ unital, associative $\mathbb{C}(q)$-algebra generated by $\{q^{\pm N}, \beta, \beta^*\}$ and

$$\begin{aligned} q^N q^{-N} &= q^{-N} q^N = 1, \quad q^N \beta = \beta q^{N-1}, \quad q^N \beta^* = \beta^* q^{N+1}, \\ \beta\beta^* - \beta^*\beta &= (1-q^2)q^{2N}, \quad \beta\beta^* - q^2 \beta^*\beta = 1 - q^2. \end{aligned}$$

Quantum Hamiltonian and Heisenberg equation [Kulish et al]:

$$-i\dot{\beta}_j = [H, \beta_j] = \beta_{j+1} q^{2N_j} + q^{2N_j} \beta_{j-1} - 2\beta_j, \qquad q^{2N_j} = 1 - \beta_j^* \beta_j$$

$$H = -\sum_{j=1}^{n} \left(\beta_{j+1}^* \beta_j + \beta_{j+1} \beta_j^* + 2N_j\right), \quad \beta_{j+n} = \beta_j, \quad q = e^{-\hbar c/2}$$

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q-bosons and Fock space

Recall the q-derivative on the space of polynomials $f \in \mathbb{C}[\zeta]$,

$$D_q f(\zeta) := \frac{f(\zeta) - f(\zeta q^2)}{\zeta - q^2 \zeta} = \sum_{r=0}^{\infty} \frac{(q^2-1)^r}{(r+1)!} \zeta^r \frac{d^{r+1}}{d\zeta^{r+1}} f(\zeta)$$

$\mathcal{F} :=$ completion of $(\mathbb{C}[\zeta], \langle \bullet | \bullet \rangle)$ with $\langle f | g \rangle = \bar{f}(D_q) g(\zeta)|_{\zeta=0}$.

$$(\beta^* f)(\zeta) = \zeta f(\zeta), \qquad \beta = (1-q^2) D_q, \qquad N = \zeta \frac{d}{d\zeta}$$

Physical interpretation

β, β^* destroy/create a q-boson, $|m\rangle$ is a state with m q-bosons.

n-fold tensor product $\mathcal{F}^{\otimes n}$ (analytic functions in n-variables):

$$(\beta_i^* f)(\zeta_1, \ldots, \zeta_n) := \zeta_i f(\zeta_1, \ldots, \zeta_n), \quad \beta_i = (1-q^2) D_{q,i}, \quad N_i = \zeta_i \frac{\partial}{\partial \zeta_i}$$

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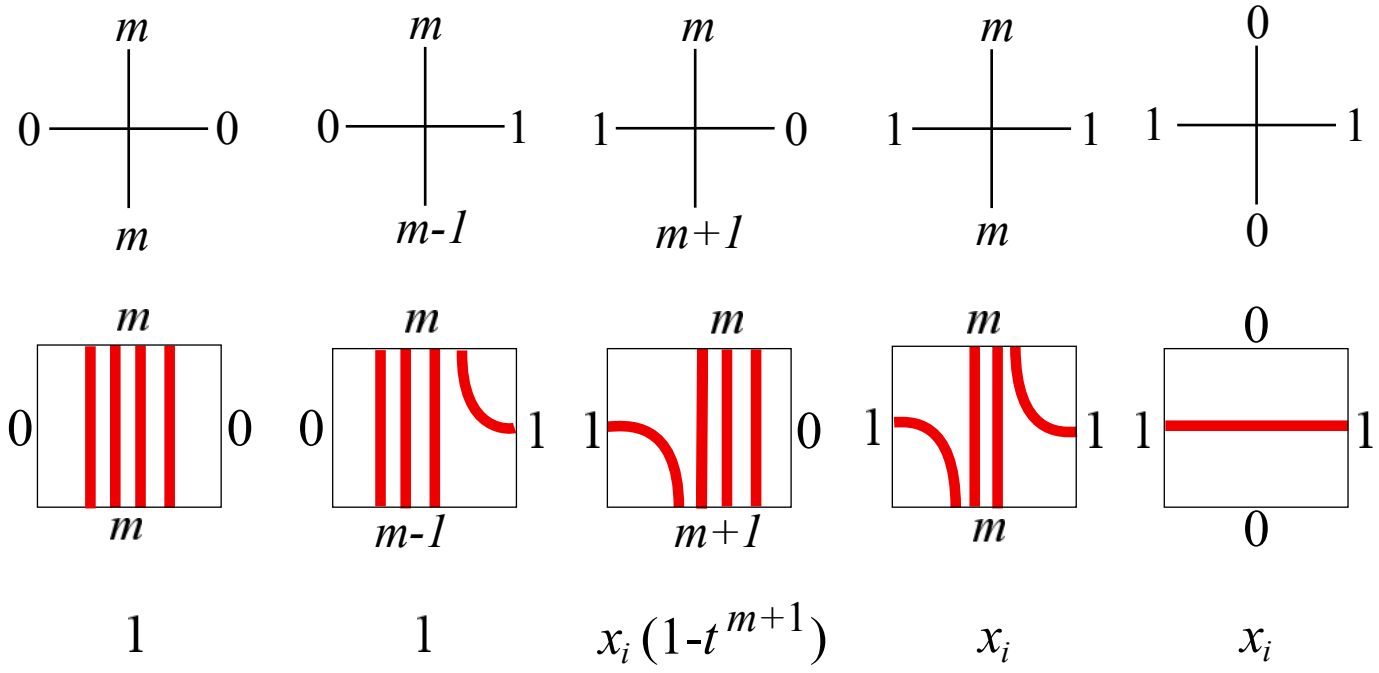
Exactly solvable lattice models: ∞-friendly walkers



Define $L_i(x) := \begin{pmatrix} 1 & x\beta_i^* \\ \beta_i & x\cdot 1 \end{pmatrix}$ and set $T(x) = L_n(x)\cdots L_1(x)$, $t = q^2$.

Proposition (Kulish, Bogoliubov-Izergin-Kitanine)

There exists a nonsingular 4×4 matrix $R = R(x,y)$ such that

$$R_{12}(x,y)T_1(x)T_2(y) = T_2(y)T_1(x)R_{12}(x,y)$$

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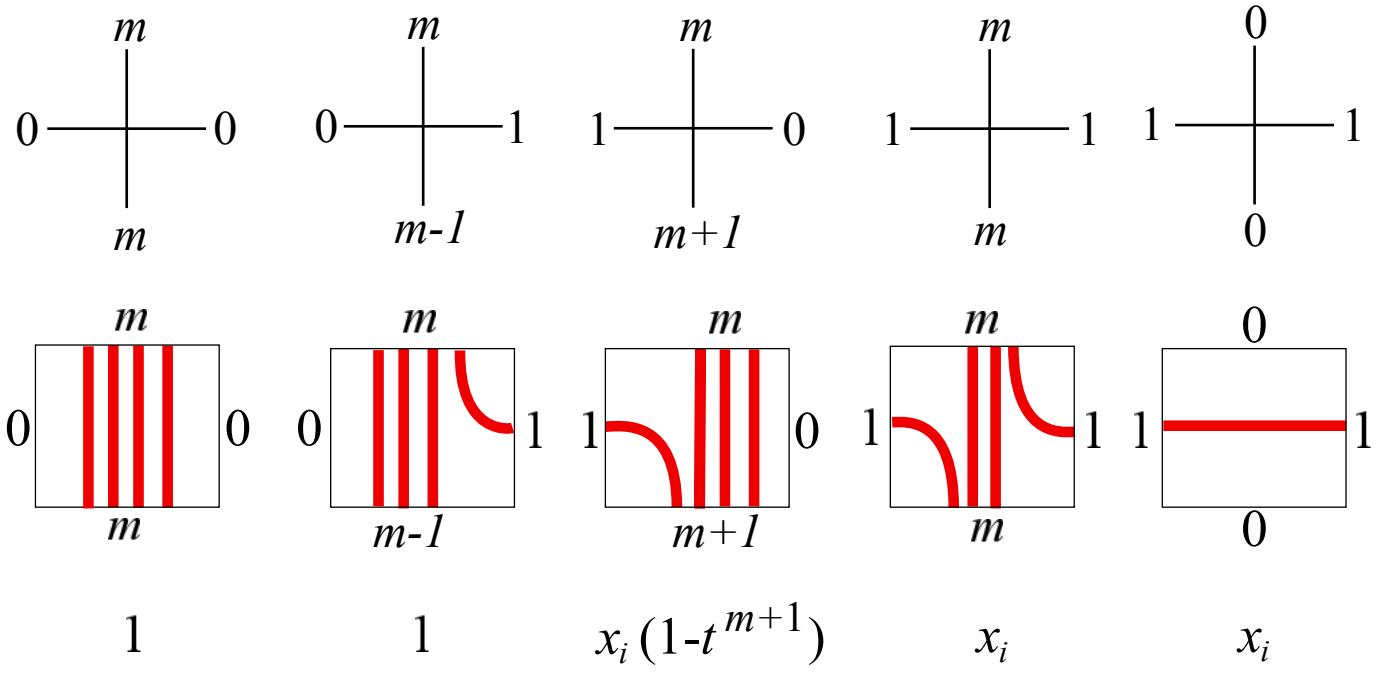

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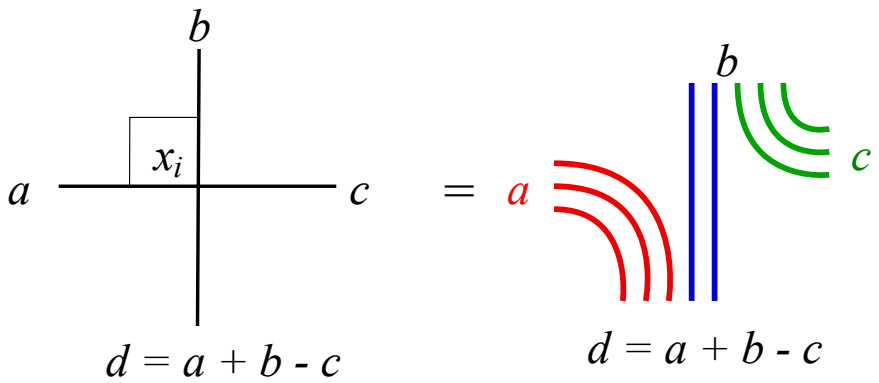
New solution to the Yang-Baxter equation (Baxter's Q)

Define another statistical mechanics model:



$$x_i^a \frac{(1-q^{2d})\cdots(1-q^{2d-a+1})}{(1-q^2)\cdots(1-q^{2a})}$$

$$L'(x) = (L'(x)_{mm'}) \text{ with } L'(x)_{mm'} = x^m(\beta^*)^m\beta^{m'} / \prod_{r=1}^{m}(1-q^{2r}) .$$

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There exists an invertible $R'(x,y)$ such that

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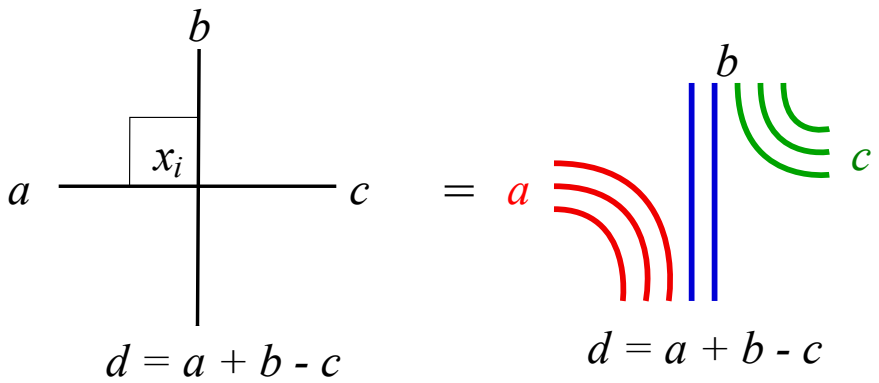

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Bethe ansatz equations

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The transfer matrices commute, $[\mathbf{T}(x), \mathbf{Q}(y)] = 0$, *and*

$$\mathbf{T}(-x)\mathbf{Q}(x) = \mathbf{Q}(xq^2) + z(-x)^n \mathbf{Q}(xq^{-2}) \prod_{i=1}^{n} q^{2N_i}$$

Consider the subspace $\mathcal{F}_k^{\otimes n} \subset \mathcal{F}^{\otimes n}$ of k walkers with k fixed.

Corollary (Bethe ansatz equations)

The roots of the eigenvalue $E(x) = \prod_{i=1}^{k}(1 + x\, y_i)$ *obey*

$$y_i^n \prod_{j \neq i} \frac{y_i - y_j t}{y_i - y_j} = \prod_{j \neq i} \frac{y_i t - y_j}{y_i - y_j}, \qquad i = 1, \ldots, k$$

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Let $E(x) = \sum_{r\geq 0} e_r(y)x^r$ and $E(-xt)/E(x) = \sum_{r\geq 0} g_r(y;t)x^r$.

Lemma (Reformulation of the BAE)

$$g_n(y;t) = 1 - t^k, \quad g_{n+r}(y;t) + t^k g_r(y;t^{-1}) = 0, \quad 0 < r < k\,.$$

Consider the ring $\mathbb{C}\{\{t\}\}[e_1,\ldots,e_k]$ where

$$\mathbb{C}\{\{t\}\} := \bigcup_{m=1}^{\infty} \mathbb{C}((t^{1/m})) \qquad \text{(Puiseux series)}.$$

Then the BAE define an algebraic variety $V_k \subset \mathbb{C}\{\{t\}\}^k$. Let $\mathcal{I} := \{f : f(y) = 0,\ y \in V_k\}$ be its nullstellen ideal.

Theorem (completeness)

$\mathfrak{R}_{n,k} := \mathbb{C}\{\{t\}\}[e_1,\ldots,e_k]/\mathcal{I}$ has dimension $\dim \mathcal{F}_k^{\otimes n}$, where $\mathcal{I} = \langle g_n(t) + t^k - 1, g_{n+1}(t) + t^k g_1(t^{-1}), \ldots, g_{n+k-1}(t) + t^k g_{k-1}(t^{-1})\rangle$.

Need to identify a suitable basis in $\mathfrak{R}_{n,k} \rightsquigarrow$ solution of BAE.

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Spherical Hecke algebra & Hall-Littlewood functions

Bernstein presentation: $\hat{H}_k$ is generated by $\{T_1, \ldots, T_{k-1}\}$ and a set of commuting, invertible elements $\{Y_1, \ldots, Y_k\}$ obeying

$$(T_i - q^{-1})(T_i + q) = 0,$$
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Define $\mathbf{1} := (1/c_k) \sum_{w \in S_k} q^{-\ell(w)} T_w$ with $c_k = \sum_w q^{-\ell(w)}$.

Satake isomorphism (spherical Macdonald functions)

Let $\Phi : \mathcal{Z}(\hat{H}_k) \longrightarrow \mathfrak{H}_k := \mathbf{1}\hat{H}_k\mathbf{1}$ be the map

$$\Phi : \quad \frac{c_\lambda}{c_k} P_\lambda(Y_1, \ldots, Y_k; t = q^2) \mapsto M_\lambda := \mathbf{1} Y^\lambda \mathbf{1}$$

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Extended affine Weyl group symmetry

$P_\lambda(t) := \prod_{i\geq 0} \frac{(1-t)^k}{(t)_{m_i(\lambda)}} R_\lambda(t)$ obeys the *straightening rules*

$$R_{\lambda.\sigma_i} = tR_\lambda - R_{(\ldots,\lambda_i-1,\lambda_{i+1}+1,\ldots)} + tR_{(\ldots,\lambda_{i+1}+1,\lambda_i-1,\ldots)} \tag{1}$$

Let $\mathcal{I}_{n,k}$ be the ideal generated by

$$R_{\lambda.\sigma_0} - tR_\lambda + R_{(\lambda_1+1,\ldots,\lambda_k-1)} - tR_{(\lambda_k-1+n,\ldots,\lambda_1+1-n)}$$
$$R_\lambda - R_{(\lambda_2,\ldots,\lambda_k,\lambda_1-n)} \tag{2}$$

Theorem (solution of BAE: deformed Verlinde algebra)

We have $\mathfrak{R}_{n,k} \cong \mathfrak{H}_k/\mathcal{I}_{n,k}$. Moreover,

- *$\{P_{\lambda'}(t)\}_{\lambda\in\mathcal{A}_{n,k}}$ forms a basis.*
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Extended affine Weyl group symmetry

$P_\lambda(t) := \prod_{i\geq 0} \frac{(1-t)^k}{(t)_{m_i(\lambda)}} R_\lambda(t)$ obeys the *straightening rules*

$$R_{\lambda.\sigma_i} = tR_\lambda - R_{(\ldots,\lambda_i-1,\lambda_{i+1}+1,\ldots)} + tR_{(\ldots,\lambda_{i+1}+1,\lambda_i-1,\ldots)} \tag{1}$$

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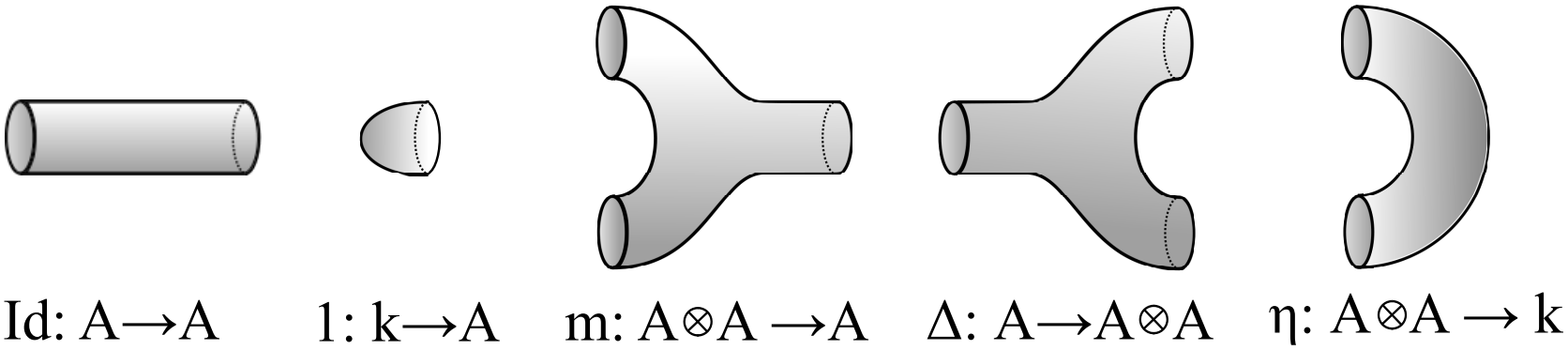
Reminder: Frobenius algebras

A Frobenius algebra A is a finite-dimn'l, unital, assoc algebra with non-degenerate bilinear form $\eta(a \cdot b, c) = \eta(a, b \cdot c),\ a, b, c \in A$.



Topological quantum field theories

Commutative Frobenius algebras are categorically equivalent to 2D topological quantum field theories.

Proposition

The Verlinde algebra is a commutative Frobenius algebra over $\mathbb{C}$ with (non-degenerate) bilinear form $\eta(\lambda, \mu) = \delta_{\lambda^* \mu}$.

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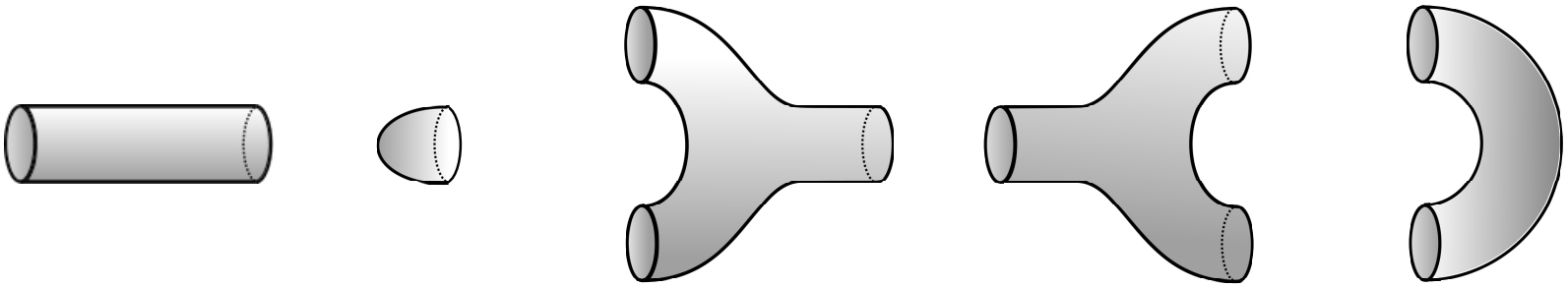

Id: A→A 1: k→A m: A⊗A →A Δ: A→A⊗A η: A⊗A → k

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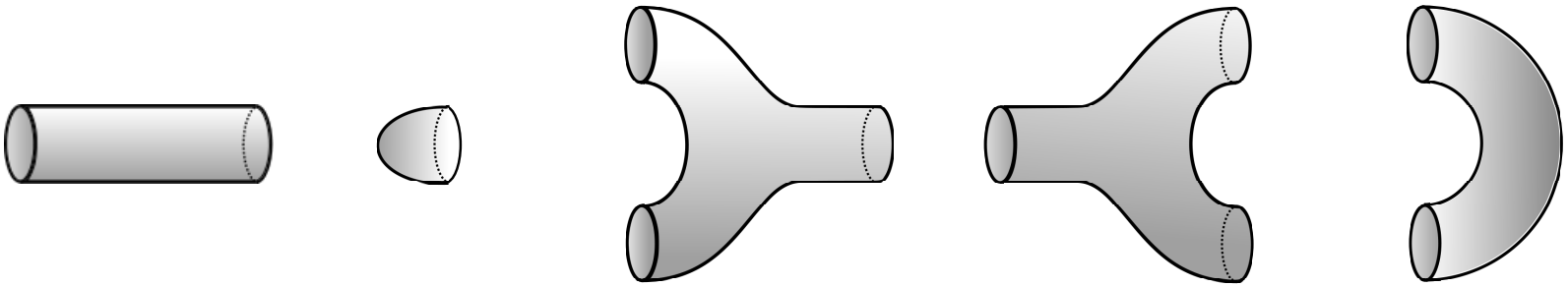

$\mathrm{Id}: A \to A$ $\quad 1: k \to A$ $\quad m: A \otimes A \to A$ $\quad \Delta: A \to A \otimes A$ $\quad \eta: A \otimes A \to k$

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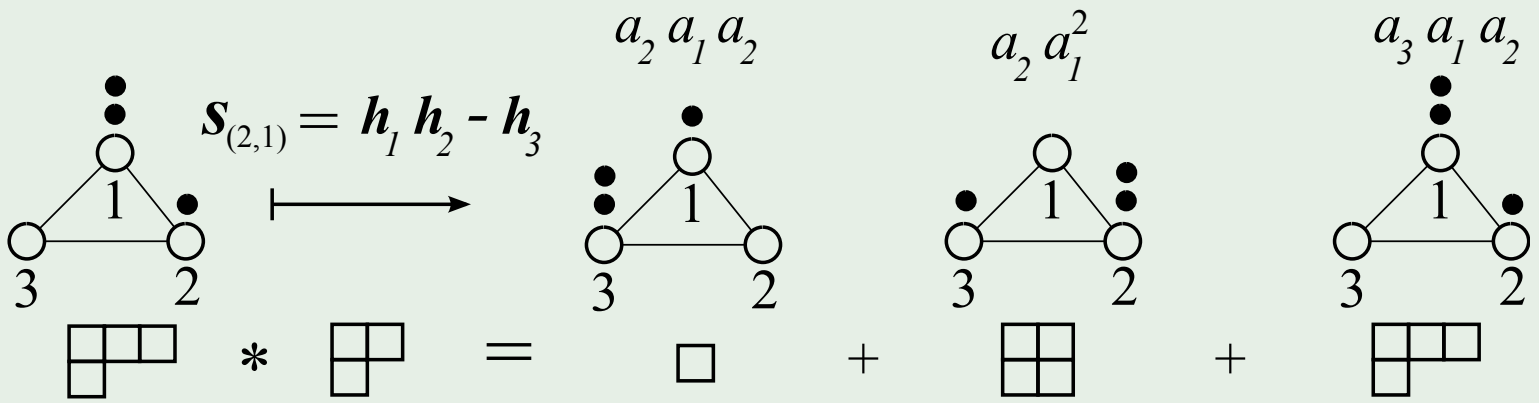
The Verlinde ring as Kirillov-Reshetikhin crystal

Example

Set $\mathfrak{g} = \mathfrak{sl}(n)$ and $n = k = 3$.



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- $\mathbf{s}_\lambda$ = Schur polynomial in Kashiwara's crystal operators a_i

Theorem (Korff-Stroppel)

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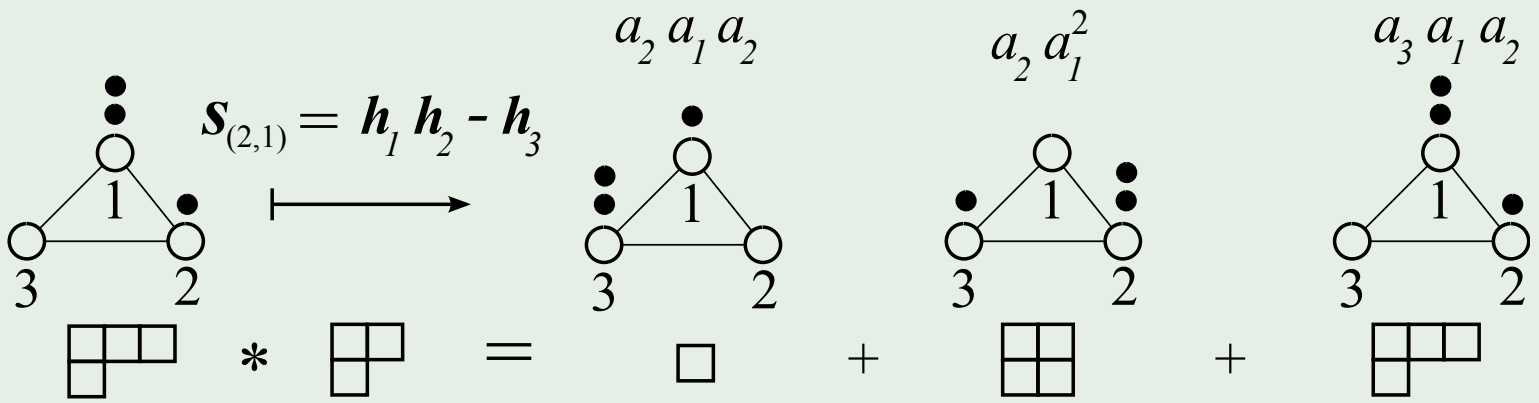

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In the context of 4D SUSY $N = 2$ Yang-Mills theories Shatashvili and Nekrasov have conjectured that the infrared limit is described by a 2D TFT whose states are in one-to-one correspondence with the eigenstates of (integrable) quantum many body systems.

- noncommutative algebra $\rightsquigarrow$ Yang-Baxter equation
- transfer matrices $\rightsquigarrow$ commutative Frobenius algebra
- Bethe ansatz equations $\rightsquigarrow$ Verlinde formula
- Baxter operator $\rightsquigarrow$ chiral ring
- Baxter equation $\rightsquigarrow$ Verlinde algebra

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Macdonald functions

Ring of symm functions: $\Lambda = \mathbb{C}[p_1, p_2, \ldots]$ with $p_r = \sum_i x_i^r$. Set

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda\mu} z_\lambda \prod_i \Big(\frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \Big), \qquad p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots$$

Theorem (Macdonald)

$\mathbb{C}(q,t) \otimes \Lambda$ has a unique orthogonal basis such that

$$P_\lambda(q,t) = m_\lambda + \sum_{\mu < \lambda} c_\mu(q,t) m_\mu$$

with m_λ monomial symmetric function.

Denote by $Q_\lambda(q,t)$ the dual basis, i.e. $\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda\mu}$

- $P_\lambda(0,t)$ – Hall-Littlewood functions
- $P_\lambda(q,0)$ – Demazure characters, q-Whittaker functions

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$$\text{Product:} \quad P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu}(q,t) P_\lambda, \quad f^\lambda_{\mu\nu}(q,t) \in \mathbb{Z}[q,t]$$

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Partition functions and cylindric skew Macdonald functions

Coproduct of Frobenius algebra (partition functions):

$$\Delta P_\lambda = \sum_{d,\mu \in \mathcal{A}_{n,k}} P_{\lambda/d/\mu} \otimes P_\mu$$

Here $P_{\lambda/d/\mu}$ is a cylindric generalisation of skew q-Whittaker functions.

$$P_{\lambda/d/\mu} = \sum_{\nu \in \mathcal{A}_{n,k}} N^\lambda_{\mu\nu}(t) P_\nu = \sum_{\nu \in \mathcal{A}_{n,k}} K_{\lambda/d/\mu,\nu}(t) s_\nu$$

Conjecture (generalised Kostka-Foulkes polynomials)

$K_{\lambda/d/\mu,\nu}(t)$ are polynomials in $t = q^2$ with non-negative coefficients.

For $\mu = (k, \ldots, k)$ these are Kostka-Foulkes polynomials.

Example

Choose $n = k = 5$ and set $\mu' = (3,2,2,1,1)$, $\nu' = (4,3,1)$.

λ'	$\langle \lambda' \vert \mathbf{S}_\nu \vert \mu' \rangle = \sum_w \varepsilon(w) K_{\lambda'/d/\mu', w(\nu+\rho)-\rho}$
$4,2,2,2,2$	$1+3q+6q^2+8q^3+8q^4+6q^5+3q^6+q^7$
$4,3,2,2,1$	$2+9q+16q^2+14q^3+6q^4+q^5$
$4,3,3,1,1$	$1+6q+13q^2+14q^3+8q^4+2q^5$
$4,4,2,1,1$	$1+6q+11q^2+10q^3+3q^4$
$2,2,1,1,1$	$1+3q+4q^2+3q^3+q^4$
$3,1,1,1,1$	$1+2q+3q^2+3q^3+2q^4+q^5$
$5,4,3,3,2$	$1+7q+20q^2+31q^3+29q^4+17q^5+6q^6+q^7$
$5,4,4,2,2$	$1+6q+17q^2+24q^3+20q^4+9q^5+2q^6$
$5,2,2,2,1$	$2+6q+9q^4+7q^3+3q^4$
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