The quantum Ablowitz-Ladik model: completeness of the Bethe ansatz

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Outline

1. q-boson (Heisenberg) algebra and quantum Ablowitz-Ladik
2. exactly solvable lattice models
3. TQ-relation and Bethe ansatz equations
4. Combinatorial solution: spherical Hecke algebra and affine straightening rules
5. Summary: from quantum many body systems to 2D TFT
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Quantum Ablowitz-Ladik equation

Classical (defocusing) Ablowitz-Ladik equation: $\psi_j \in \mathbb{C}, \ |\psi_j| < 1$

$$-i\dot{\psi}_j = (1 - |\psi_j|^2)(\psi_{j+1} + \psi_{j-1}) - 2\psi_j, \quad \psi_{j+n} = \psi_j$$

Definition ($q$-deformed boson algebra)

$\mathcal{H}_q$ unital, associative $\mathbb{C}(q)$-algebra generated by $\{q^{\pm N}, \beta, \beta^*\}$ and

$$q^N q^{-N} = q^{-N} q^N = 1, \quad q^N \beta = \beta q^{N-1}, \quad q^N \beta^* = \beta^* q^{N+1},$$

$$\beta \beta^* - \beta^* \beta = (1 - q^2) q^{2N}, \quad \beta \beta^* - q^2 \beta^* \beta = 1 - q^2.$$  

Quantum Hamiltonian and Heisenberg equation [Kulish et al.]:

$$-i\dot{\beta}_j = [H, \beta_j] = \beta_{j+1} q^{2N_j} + q^{2N_j} \beta_{j-1} - 2\beta_j, \quad q^{2N_j} = 1 - \beta_j^* \beta_j$$

$$H = -\sum_{j=1}^{n} (\beta_{j+1}^* \beta_j + \beta_{j+1} \beta_j^* + 2N_j), \quad \beta_{j+n} = \beta_j, \quad q = e^{-\hbar c/2}$$
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Recall the $q$-derivative on the space of polynomials $f \in \mathbb{C}[\zeta]$,

$$D_q f(\zeta) := \frac{f(\zeta) - f(\zeta q^2)}{\zeta - q^2 \zeta} = \sum_{r=0}^{\infty} (q^2 - 1)^r \frac{d^{r+1}}{d\zeta^{r+1}} f(\zeta) \left(\frac{\zeta}{(r+1)!} \right)$$

$\mathcal{F} :=$ completion of $(\mathbb{C}[\zeta], \langle \cdot | \cdot \rangle)$ with $\langle f | g \rangle = \bar{f}(D_q)g(\zeta)|_{\zeta=0}$.

$$(\beta^* f)(\zeta) = \zeta f(\zeta), \quad \beta = (1 - q^2)D_q, \quad N = \zeta \frac{d}{d\zeta}$$

Physical interpretation

$\beta$, $\beta^*$ destroy/create a $q$-boson, $|m\rangle$ is a state with $m$ $q$-bosons.

$n$-fold tensor product $\mathcal{F}^{\otimes n}$ (analytic functions in $n$-variables):

$$(\beta_i^* f)(\zeta_1, \ldots, \zeta_n) := \zeta_i f(\zeta_1, \ldots, \zeta_n), \quad \beta_i = (1 - q^2)D_{q,i}, \quad N_i = \zeta_i \frac{\partial}{\partial \zeta_i}$$
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Define $L_i(x) := \left( \frac{1}{\beta_i} x \beta_i^* \right)$ and set $T(x) = L_n(x) \cdots L_1(x)$, $t = q^2$.

**Proposition (Kulish, Bogoliubov-Izergin-Kitanine)**

There exists a nonsingular $4 \times 4$ matrix $R = R(x, y)$ such that

$$R_{12}(x, y) T_1(x) T_2(y) = T_2(y) T_1(x) R_{12}(x, y)$$
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Define another statistical mechanics model:

\[ d = a + b - c \]

\[ (1-q^{2d}) \cdots (1-q^{2d-a+1}) \]

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\[ L'(x) = (L'(x)_{mm'}) \text{ with } L'(x)_{mm'} = x^m (\beta^*)^m \beta^{m'} / \prod_{r=1}^{m} (1 - q^{2r}) . \]

**Proposition (Yang-Baxter equation)**

There exists an invertible \( R'(x, y) \) such that

\[ R'_{12}(x, y)L'_1(x)L'_2(y) = L'_2(y)L'_1(x)R'_{12}(x, y) . \]
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\[\Rightarrow \text{ QNLS discrete Laplacians [van Diejen]}\]
Proposition (analogue of Baxter’s TQ-equation)

The transfer matrices commute, \([T(x), Q(y)] = 0\), and

\[
T(-x)Q(x) = Q(xq^2) + z(-x)^n Q(xq^{-2}) \prod_{i=1}^{n} q^{2N_i}
\]

Consider the subspace \(F_k \otimes \mathbb{C}^n \subset F \otimes \mathbb{C}^n\) of \(k\)walkers with \(k\) fixed.

Corollary (Bethe ansatz equations)

The roots of the eigenvalue \(E(x) = \prod_{i=1}^{k} (1 + x y_i)\) obey

\[
y_i^n \prod_{j \neq i} \frac{y_i - y_j t}{y_i - y_j} = \prod_{j \neq i} \frac{y_i t - y_j}{y_i - y_j}, \quad i = 1, \ldots, k
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\]
Let $E(x) = \sum_{r \geq 0} e_r(y)x^r$ and $E(-xt)/E(x) = \sum_{r \geq 0} g_r(y; t)x^r$.

**Lemma (Reformulation of the BAE)**

$g_n(y; t) = 1 - t^k$, \hspace{1em} $g_{n+r}(y; t) + t^k g_r(y; t^{-1}) = 0$, \hspace{1em} $0 < r < k$.

Consider the ring $\mathbb{C}\{\{t\}\}[e_1, \ldots, e_k]$ where

$$\mathbb{C}\{\{t\}\} := \bigcup_{m=1}^{\infty} \mathbb{C}((t^{1/m})) \quad \text{(Puiseux series).}$$

Then the BAE define an algebraic variety $V_k \subset \mathbb{C}\{\{t\}\}^k$. Let $\mathcal{I} := \{f : f(y) = 0, \ y \in V_k\}$ be its nullstellen ideal.

**Theorem (completeness)**

$\mathcal{R}_{n,k} := \mathbb{C}\{\{t\}\}[e_1, \ldots, e_k]/\mathcal{I}$ has dimension $\dim F_k^\otimes n$, where $\mathcal{I} = \langle g_n(t) + t^k - 1, g_{n+1}(t) + t^k g_1(t^{-1}), \ldots, g_{n+k-1}(t) + t^k g_{k-1}(t^{-1}) \rangle$.

Need to identify a suitable basis in $\mathcal{R}_{n,k} \hookrightarrow$ solution of BAE.
Let $E(x) = \sum_{r \geq 0} e_r(y)x^r$ and $E(-xt)/E(x) = \sum_{r \geq 0} g_r(y;t)x^r$.

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Bernstein presentation: $\hat{H}_k$ is generated by $\{T_1, \ldots, T_{k-1}\}$ and a set of commuting, invertible elements $\{Y_1, \ldots, Y_k\}$ obeying

$$(T_i - q^{-1})(T_i + q) = 0,$$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \quad |i - j| > 1, \quad i, j \in \mathbb{Z}_k$

$T_i Y_i T_i = Y_{i+1}, \quad T_i Y_j = Y_j T_i \quad \text{for} \quad j \neq i, i + 1$

Define $1 := (1/c_k) \sum_{w \in S_k} q^{-\ell(w)} T_w$ with $c_k = \sum_w q^{-\ell(w)}$.

Satake isomorphism (spherical Macdonald functions)

Let $\Phi : \mathcal{Z}(\hat{H}_k) \rightarrow \mathcal{H}_k := 1\hat{H}_k 1$ be the map

$$\Phi : \frac{c^\lambda}{c_k} P_\lambda(Y_1, \ldots, Y_k; t = q^2) \mapsto M_\lambda := 1 Y^\lambda 1$$

where $\{M_\lambda : \lambda_1 \geq \ldots \geq \lambda_k \geq 0\}$ is a basis of $\mathcal{H}_k$. 

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Extended affine Weyl group symmetry

\[ P_\lambda(t) := \prod_{i \geq 0} \frac{(1-t)^k}{(t)^{m_i(\lambda)}} R_\lambda(t) \] obeys the straightening rules

\[ R_{\lambda, \sigma_i} = tR_\lambda - R(\ldots, \lambda_{i-1}, \lambda_{i+1}+1, \ldots) + tR(\ldots, \lambda_{i+1}+1, \lambda_{i-1}, \ldots) \] (1)

Let \( I_{n,k} \) be the ideal generated by

\[ R_{\lambda, \sigma_0} - tR_\lambda + R(\lambda_1+1, \ldots, \lambda_{k-1}) - tR(\lambda_{k-1}+n, \ldots, \lambda_1+1-n) \]

\[ R_\lambda - R(\lambda_2, \ldots, \lambda_k, \lambda_1-n) \] (2)

Theorem (solution of BAE: deformed Verlinde algebra)

We have \( \mathcal{R}_{n,k} \cong \mathcal{H}_k / I_{n,k} \). Moreover,

- \( \{ P_{\lambda'}(t) \}_{\lambda \in A_{n,k}} \) forms a basis.
- \( \mathcal{R}_{n,k} \) is a commutative Frobenius algebra (2D TFT).
- Strong coupling limit \( t = 0 \): \( \mathcal{R}_{n,k} / \langle t \rangle \cong \text{Verlinde algebra} \).

\[ P_{\lambda'}(t)P_{\mu'}(t) = \sum_{\nu \in A_{n,k}} N_{\lambda \mu}^\nu(t) P_{\nu'}(t), \quad N_{\lambda \mu}^\nu(0) \text{ WZNW fusion coefficient} \]
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\[ R_{\lambda,\sigma_i} = tR_\lambda - R(...,\lambda_i-1,\lambda_{i+1}+1,...) + tR(...,\lambda_{i+1}+1,\lambda_i-1,...) \tag{1} \]

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\[ R_{\lambda,\sigma_0} - tR_\lambda + R(\lambda_1+1,...,\lambda_k-1) - tR(\lambda_k-1+n,...,\lambda_1+1-n) \]

\[ R_\lambda - R(\lambda_2,...,\lambda_k,\lambda_1-n) \tag{2} \]

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\[ P_{\lambda'}(t)P_{\mu'}(t) = \sum_{\nu \in A_{n,k}} N_{\lambda\mu}^\nu(t)P_{\nu'}(t), \quad N_{\lambda\mu}^\nu(0) \text{ WZNW fusion coefficient} \]
Extended affine Weyl group symmetry

\[ P_\lambda(t) := \prod_{i \geq 0} \frac{(1-t)^k}{(t)^{m_i(\lambda)}} R_\lambda(t) \] obeys the straightening rules

\[ R_{\lambda, \sigma_i} = tR_\lambda - R(\ldots, \lambda_i - 1, \lambda_{i+1} + 1, \ldots) + tR(\ldots, \lambda_{i+1} + 1, \lambda_i - 1, \ldots) \tag{1} \]

Let \( I_{n,k} \) be the ideal generated by

\[ R_{\lambda, \sigma_0} - tR_\lambda + R(\lambda_1 + 1, \ldots, \lambda_k - 1) - tR(\lambda_k - 1 + n, \ldots, \lambda_1 + 1 - n) \]

\[ R_\lambda - R(\lambda_2, \ldots, \lambda_k, \lambda_1 - n) \tag{2} \]

Theorem (solution of BAE: deformed Verlinde algebra)

We have \( \mathcal{R}_{n,k} \cong \mathfrak{H}_k / I_{n,k} \). Moreover,

- \( \{P_{\lambda'}(t)\}_{\lambda \in \mathcal{A}_{n,k}} \) forms a basis.
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Let \( \mathcal{I}_{n,k} \) be the ideal generated by

\[ R_{\lambda, \sigma_0} - tR_\lambda + R(\lambda_1+1, \ldots, \lambda_{k-1}) - tR(\lambda_{k-1}+n, \ldots, \lambda_1+1-n) \]

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A Frobenius algebra $A$ is a finite-dimn’l, unital, assoc algebra with non-degenerate bilinear form $\eta(a \cdot b, c) = \eta(a, b \cdot c)$, $a, b, c \in A$.

Commutative Frobenius algebras are categorically equivalent to 2D topological quantum field theories.

Proposition

The Verlinde algebra is a commutative Frobenius algebra over $\mathbb{C}$ with (non-degenerate) bilinear form $\eta(\lambda, \mu) = \delta_{\lambda^* \mu}$.
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**Topological quantum field theories**

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Christian Korff  The quantum Ablowitz-Ladik model
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The Verlinde ring as Kirillov-Reshetikhin crystal

Example

Set $g = sl(n)$ and $n = k = 3$.

$\begin{align*}
S_{(2,1)} &= h_1 h_2 - h_3 \\
a_2 a_1 a_2 &= a_2^2 a_1 \\
a_3 a_1 a_2
\end{align*}$

- Particle configuration = Young diagram $\in B_{k\omega_1}$ (KR crystal)
- $s_\lambda$ = Schur polynomial in Kashiwara’s crystal operators $a_i$

Theorem (Korff-Stroppel)

$\mathbb{Z}P^+_k$ with product $\lambda \boxtimes \mu := s_\lambda \mu$ is canonically isomorphic to $\mathcal{V}_k$. 

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\[
\begin{align*}
\mathbf{s}_{(2,1)} &= h_1 h_2 - h_3 \\
1 &\quad 3 \\
2 &\quad 3 \\
\end{align*}
\]

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In the context of 4D SUSY $N = 2$ Yang-Mills theories Shatashvili and Nekrasov have conjectured that the infrared limit is described by a 2D TFT whose states are in one-to-one correspondence with the eigenstates of (integrable) quantum many body systems.

- noncommutative algebra $\mapsto$ Yang-Baxter equation
- transfer matrices $\mapsto$ commutative Frobenius algebra
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Macdonald functions

Ring of symm functions: \( \Lambda = \mathbb{C}[p_1, p_2, \ldots] \) with \( p_r = \sum_i x_i^r \). Set

\[
\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda\mu} z_\lambda \prod_i \left( \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \right), \quad p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots
\]

**Theorem (Macdonald)**

\( \mathbb{C}(q,t) \otimes \Lambda \) has a unique orthogonal basis such that

\[
P_\lambda(q,t) = m_\lambda + \sum_{\mu < \lambda} c_\mu(q,t) m_\mu
\]

with \( m_\lambda \) monomial symmetric function.

Denote by \( Q_\lambda(q,t) \) the dual basis, i.e. \( \langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda\mu} \)

- \( P_\lambda(0,t) \) – Hall-Littlewood functions
- \( P_\lambda(q,0) \) – Demazure characters, q-Whittaker functions

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Skew Macdonald functions

The algebra of symmetric functions is naturally endowed with a product $\Lambda \otimes \Lambda \to \Lambda$.

The adjoint operation w.r.t. the inner product leads to the co-product $\Lambda \to \Lambda \otimes \Lambda$,

$$\langle f, g \cdot h \rangle_{q,t} =: \langle \Delta f, g \otimes h \rangle_{q,t}, \quad f, g, h \in \mathbb{C}(q, t) \otimes \Lambda$$

Co-algebra structure

Product: $P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu}(q, t) P_\lambda, \quad f^\lambda_{\mu\nu}(q, t) \in \mathbb{Z}[q, t]$

coproduct: $\Delta P_\lambda = \sum_\mu P_{\lambda/\mu} \otimes P_\mu, \quad P_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu}(q, t) P_\nu$

Skew Hall-Littlewood functions ($q = 0$): $\Delta Q_\lambda = \sum_\mu Q_{\lambda/\mu} \otimes Q_\mu$
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Partition functions and cylindric skew Macdonald functions

Coproduct of Frobenius algebra (partition functions):

\[ \Delta P_{\lambda} = \sum_{d, \mu \in \mathcal{A}_{n,k}} P_{\lambda/d/\mu} \otimes P_{\mu} \]

Here \( P_{\lambda/d/\mu} \) is a cylindric generalisation of skew q-Whittaker functions.

\[ P_{\lambda/d/\mu} = \sum_{\nu \in \mathcal{A}_{n,k}} N_{\mu \nu}^\lambda(t) P_{\nu} = \sum_{\nu \in \mathcal{A}_{n,k}} K_{\lambda/d/\mu,\nu}(t)s_{\nu} \]

Conjecture (generalised Kostka-Foulkes polynomials)

\( K_{\lambda/d/\mu,\nu}(t) \) are polynomials in \( t = q^2 \) with non-negative coefficients.

For \( \mu = (k, \ldots, k) \) these are Kostka-Foulkes polynomials.
Choose $n = k = 5$ and set $\mu' = (3, 2, 2, 1, 1)$, $\nu' = (4, 3, 1)$.

| $\lambda'$ | $\langle \lambda' | S_\nu | \mu' \rangle = \sum_w \varepsilon(w) K_{\lambda'/d/\mu',w(\nu+\rho)-\rho}$ |
|------------|--------------------------------------------------|
| 4, 2, 2, 2, 2 | $1 + 3q + 6q^2 + 8q^3 + 8q^4 + 6q^5 + 3q^6 + q^7$ |
| 4, 3, 2, 2, 1 | $2 + 9q + 16q^2 + 14q^3 + 6q^4 + q^5$ |
| 4, 3, 3, 1, 1 | $1 + 6q + 13q^2 + 14q^3 + 8q^4 + 2q^5$ |
| 4, 4, 2, 1, 1 | $1 + 6q + 11q^2 + 10q^3 + 3q^4$ |
| 2, 2, 1, 1, 1 | $1 + 3q + 4q^2 + 3q^3 + q^4$ |
| 3, 1, 1, 1, 1 | $1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5$ |
| 5, 4, 3, 3, 2 | $1 + 7q + 20q^2 + 31q^3 + 29q^4 + 17q^5 + 6q^6 + q^7$ |
| 5, 4, 4, 2, 2 | $1 + 6q + 17q^2 + 24q^3 + 20q^4 + 9q^5 + 2q^6$ |
| 5, 2, 2, 2, 1 | $2 + 6q + 9q^4 + 7q^3 + 3q^4$ |
| 5, 3, 2, 1, 1 | $3 + 8q + 9q^4 + 3q^3$ |
| 5, 4, 1, 1, 1 | $1 + 3q + 4q^2 + 3q^3 + q^4$ |
| 5, 5, 3, 2, 2 | $2 + 8q + 16q^2 + 17q^3 + 10q^4 + 3q^5$ |
| 5, 5, 3, 3, 1 | $1 + 5q + 10q^2 + 10q^3 + 5q^4 + q^5$ |
| 5, 5, 4, 2, 1 | $1 + 5q + 8q^2 + 6q^3 + 2q^4$ |