Correlation functions for integrable higher spin $su(2)$ quantum chains

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Contents

- spin-1/2 Heisenberg chain

  factorization of equal time correlators

  quantum chain $\leftrightarrow$ 2d vertex model

  inhomogeneous systems on semi-infinite cylinders

  (i) $L = \infty, T \geq 0$ and (ii) mirror model: L finite, $T = 0$

  NLIE, Y-system, entanglement entropies,...

- spin-S Heisenberg chain (spin-1 case)

  fusion procedure for R-matrices

  explicit calculation for two-site reduced density matrix

  functional equations, analyticity conditions

  factorization, general case

collaborators: B. Aufgebauer, H. Boos, F. Göhmann, D. Nawrath, J. Suzuki

Support by Volkswagen Foundation

Integrable $su(2)$ quantum chains: $S = 1/2$ case

Heisenberg $S = 1/2$ chain

$$H = \sum_j \vec{S}_j \vec{S}_{j+1}$$

Near-neighbor correlators ($L = \infty$, $T = 0$):

$$\begin{aligned}
\left\langle S^z_j S^z_{j+1} \right\rangle &= \frac{1}{12} - \frac{1}{3}\ln 2 = -0.1477157268\ldots \\
\left\langle S^z_j S^z_{j+2} \right\rangle &= \frac{1}{12} - \frac{4}{3}\ln 2 + \frac{3}{4}\zeta(3) = 0.06067976995\ldots \\
\left\langle S^z_j S^z_{j+3} \right\rangle &= \frac{1}{12} - 3\ln 2 + \frac{37}{6}\zeta(3) - \frac{14}{3}\ln 2 \cdot \zeta(3) - \frac{3}{2}\zeta(3)^2 - \frac{125}{24}\zeta(5) + \frac{25}{3}\ln 2 \cdot \zeta(5) \\
&= -0.05024862725\ldots
\end{aligned}$$

Hulthén (1938), Takahashi (1977): ground state energies of Heisenberg and Hubbard model

Multiple integral formulas for density matrix:

Vertex operator approach: Jimbo, Miki, Miwa, Nakayashiki (92); qKZ equation Jimbo, Miwa (96)

Boos, Korepin (01/02)

$T = 0$, $h \geq 0$ Kitanine, Maillet, Terras (1998); $T \geq 0$, $h \geq 0$ Göhmann, AK, Seel (2004).

$su(2)$ $S = 1/2$: 2-point correlators

- analytic results show feasibility of exact calculations
- the smaller the numerical results the larger the analytical expressions

$$
\begin{aligned}
\left\langle S_j^z S_{j+4}^z \right\rangle &= \frac{1}{12} - \frac{16}{3}\ln 2 + \frac{145}{6}\zeta(3) - 54\ln 2\cdot\zeta(3) - \frac{293}{4}\zeta(3)^2 - \frac{875}{12}\zeta(5) + \frac{1450}{3}\ln 2\cdot\zeta(5) \\
&\quad - \frac{275}{16}\zeta(3)\cdot\zeta(5) - \frac{1875}{16}\zeta(5)^2 + \frac{3185}{64}\zeta(7) - \frac{1715}{4}\ln 2\cdot\zeta(7) + \frac{6615}{32}\zeta(3)\cdot\zeta(7) \\
&= 0.034652776982\ldots \\
\left\langle S_j^z S_{j+5}^z \right\rangle &= \text{7 lines} \\
\left\langle S_j^z S_{j+6}^z \right\rangle &= \text{18 lines (= 1 page)} \\
\left\langle S_j^z S_{j+7}^z \right\rangle &= \text{3 pages}
\end{aligned}
$$

$su(2)$ $S=1/2$: more correlators

- occurence of zeta-function values seem to indicate that no algebraic structure exists

  no algebraic relation of 2-point functions for different separation

  no algebraic, *Wick theorem*-like relation of general n-site correlations

$$
\begin{aligned}
\left\langle S_j^z S_{j+1}^z \right\rangle &= \frac{1}{12} - \frac{1}{3}\ln 2 \\
\left\langle S_j^z S_{j+2}^z \right\rangle &= \frac{1}{12} - \frac{4}{3}\ln 2 + \frac{3}{4}\zeta(3) \\
\left\langle S_j^z S_{j+3}^z \right\rangle &= \frac{1}{12} - 3\ln 2 + \frac{37}{6}\zeta(3) - \frac{14}{3}\ln 2\cdot\zeta(3) - \frac{3}{2}\zeta(3)^2 - \frac{125}{24}\zeta(5) + \frac{25}{3}\ln 2\cdot\zeta(5) \\
\left\langle S_j^x S_{j+1}^x S_{j+2}^z S_{j+3}^z \right\rangle &= \frac{1}{240} + \frac{\ln 2}{12} - \frac{91}{240}\zeta(3) + \frac{1}{6}\ln 2\cdot\zeta(3) + \frac{3}{80}\zeta(3)^2 + \frac{35}{96}\zeta(5) - \frac{5}{24}\ln 2\cdot\zeta(5), \\
\left\langle S_j^x S_{j+1}^z S_{j+2}^x S_{j+3}^z \right\rangle &= \frac{1}{240} - \frac{\ln 2}{6} + \frac{77}{120}\zeta(3) - \frac{5}{12}\ln 2\cdot\zeta(3) - \frac{3}{20}\zeta(3)^2 - \frac{65}{96}\zeta(5) + \frac{5}{6}\ln 2\cdot\zeta(5), \\
\left\langle S_j^x S_{j+1}^z S_{j+2}^z S_{j+3}^x \right\rangle &= \frac{1}{240} - \frac{\ln 2}{4} + \frac{169}{240}\zeta(3) - \frac{5}{12}\ln 2\cdot\zeta(3) - \frac{3}{20}\zeta(3)^2 - \frac{65}{96}\zeta(5) + \frac{5}{6}\ln 2\cdot\zeta(5)
\end{aligned}
$$

$su(2)$ $S=1/2$: correlators of inhomogeneous system

Another example of correlators: emptiness formation probability (Boos, Korepin, Smirnov 03)

$$P(n) := \left\langle \left(S_1^z + \frac{1}{2}\right)\left(S_2^z + \frac{1}{2}\right)\cdots\left(S_n^z + \frac{1}{2}\right)\right\rangle = (D_n)^{++\dots+}_{++\dots+}$$

Explicitly for spin-1/2 Heisenberg ($T=0$)

$$\begin{aligned} P(1) &= \frac{1}{2}, \qquad P(2) = \frac{1}{3} - \frac{1}{3}\ln 2, \qquad P(3) = \frac{1}{4} - \ln 2 + \frac{3}{8}\zeta(3) \\ P(4) &= \frac{1}{5} - 2\ln 2 + \frac{173}{60}\zeta(3) - \frac{11}{6}\ln 2\cdot\zeta(3) - \frac{51}{80}\zeta^2(3) - \frac{55}{24}\zeta(5) + \frac{85}{24}\ln 2\cdot\zeta(5). \end{aligned}$$

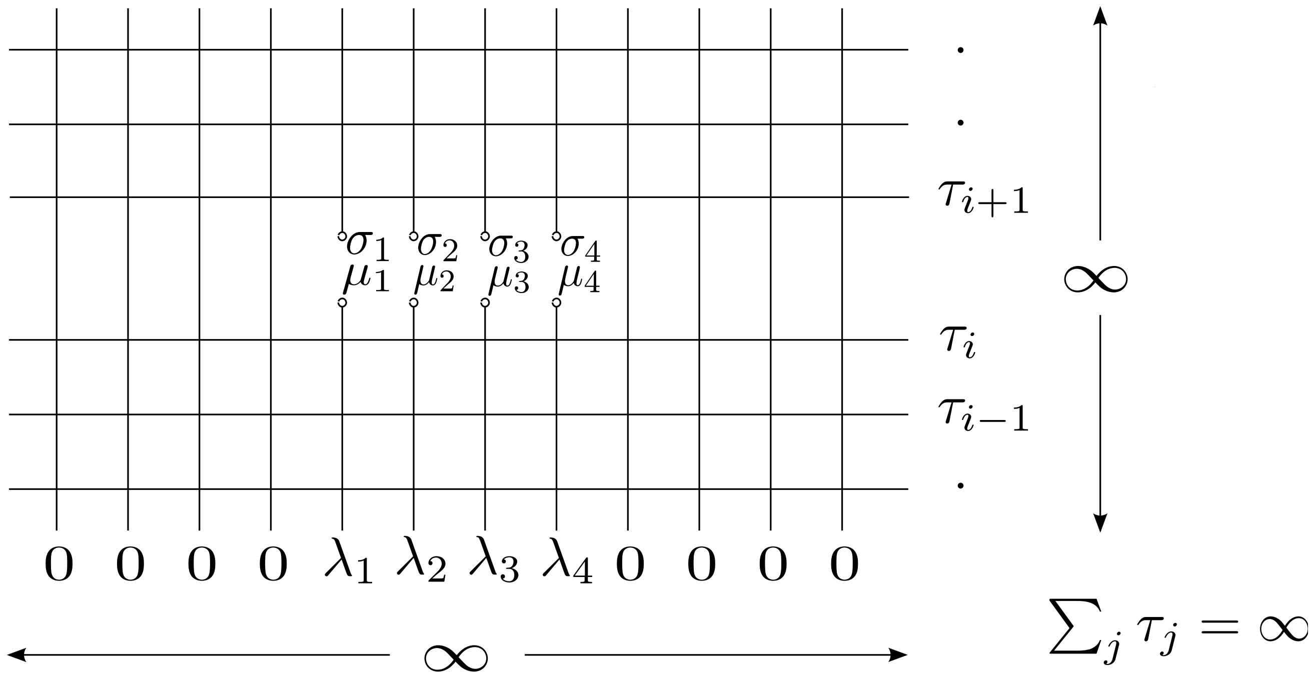
Inhomogeneous generalization of correlators enjoys more algebraic relations ($\lambda_{ij} := \lambda_i - \lambda_j$)

$$P(1) = \frac{1}{2}, \quad P(2) = \frac{1}{4} + \frac{1}{6}\omega(\lambda_1,\lambda_2), \quad P(3) = \frac{1}{8} + \frac{1+\lambda_{13}\lambda_{23}}{12\lambda_{13}\lambda_{23}}\omega(\lambda_1,\lambda_2) \quad \text{(+ 2 permutations)}$$

$$\begin{aligned} P(4) &= \frac{1}{16} + \frac{5(1+\lambda_{14}\lambda_{24})(1+\lambda_{13}\lambda_{23}) - (\lambda_{12}^2 - 4)}{120\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1,\lambda_2) \quad \text{(+ 5 permutations)} \\ &+ \frac{15 + 10(\lambda_{13}\lambda_{24}+1)(1+\lambda_{14}\lambda_{23}) + (2\lambda_{12}^2 - 3)(2\lambda_{34}^2 - 3)}{360\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1,\lambda_2)\omega(\lambda_3,\lambda_4) \\ &\text{(+ 2 permutations)} \end{aligned}$$

Mapping 1d quantum system to 2d classical allows for generalization of D_n as meromorphic function $D_n(\lambda_1, \lambda_2, ..., \lambda_n)$.

Unnormalized matrix element $D^{\mu_1,...,\mu_4}_{\sigma_1,...,\sigma_4}$ given by partition function of some six-vertex model:



The $\omega(\lambda_1, \lambda_2)$ function is given by the nearest-neighbour correlator

$$(T = h = 0) \qquad \omega(\lambda_1, \lambda_2) := \frac{1}{2} + 2\sum_{k=1}^{\infty}(-1)^k k \frac{1-\lambda^2}{k^2-\lambda^2}, \qquad \lambda := \lambda_1 - \lambda_2.$$

$su(2)$ $S = 1/2$: factorization

Various questions arise when looking at expressions for correlators like ($\lambda_{ij} := \lambda_i - \lambda_j$)

$$P(1) = \frac{1}{2}, \quad P(2) = \frac{1}{4} + \frac{1}{6}\omega(\lambda_1,\lambda_2), \quad P(3) = \frac{1}{8} + \frac{1+\lambda_{13}\lambda_{23}}{12\lambda_{13}\lambda_{23}}\omega(\lambda_1,\lambda_2) \quad \text{(+ 2 permutations)}$$

$$P(4) = \frac{1}{16} + \frac{5(1+\lambda_{14}\lambda_{24})(1+\lambda_{13}\lambda_{23}) - (\lambda_{12}^2 - 4)}{120\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1,\lambda_2) \quad \text{(+ 5 permutations)}$$
$$+ \frac{15 + 10(\lambda_{13}\lambda_{24}+1)(1+\lambda_{14}\lambda_{23}) + (2\lambda_{12}^2 - 3)(2\lambda_{34}^2 - 3)}{360\lambda_{13}\lambda_{23}\lambda_{14}\lambda_{24}}\omega(\lambda_1,\lambda_2)\omega(\lambda_3,\lambda_4)$$
$$\text{(+ 2 permutations)}$$

- How do these expressions appear? (Multiple integral expressions, functional equations)
- Is the limit $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \to 0$ singular?

  algebraically: yes $\longrightarrow$ therefore no algebraic structure in homogeneous limit

  analytically: no $\longrightarrow$ therefore nothing wrong about it
- Why should we care about such factorized expressions for correlators of inhomogenous systems?

$su(2)$ $S=1/2$: Finite temperatures

• Why should we care about such factorized expressions for correlators of inhomogenous systems?

We care, because we can prove the (literally) same factorization for arbitrary temperature...

... and we can calculate $\omega(\lambda_1,\lambda_2)$!

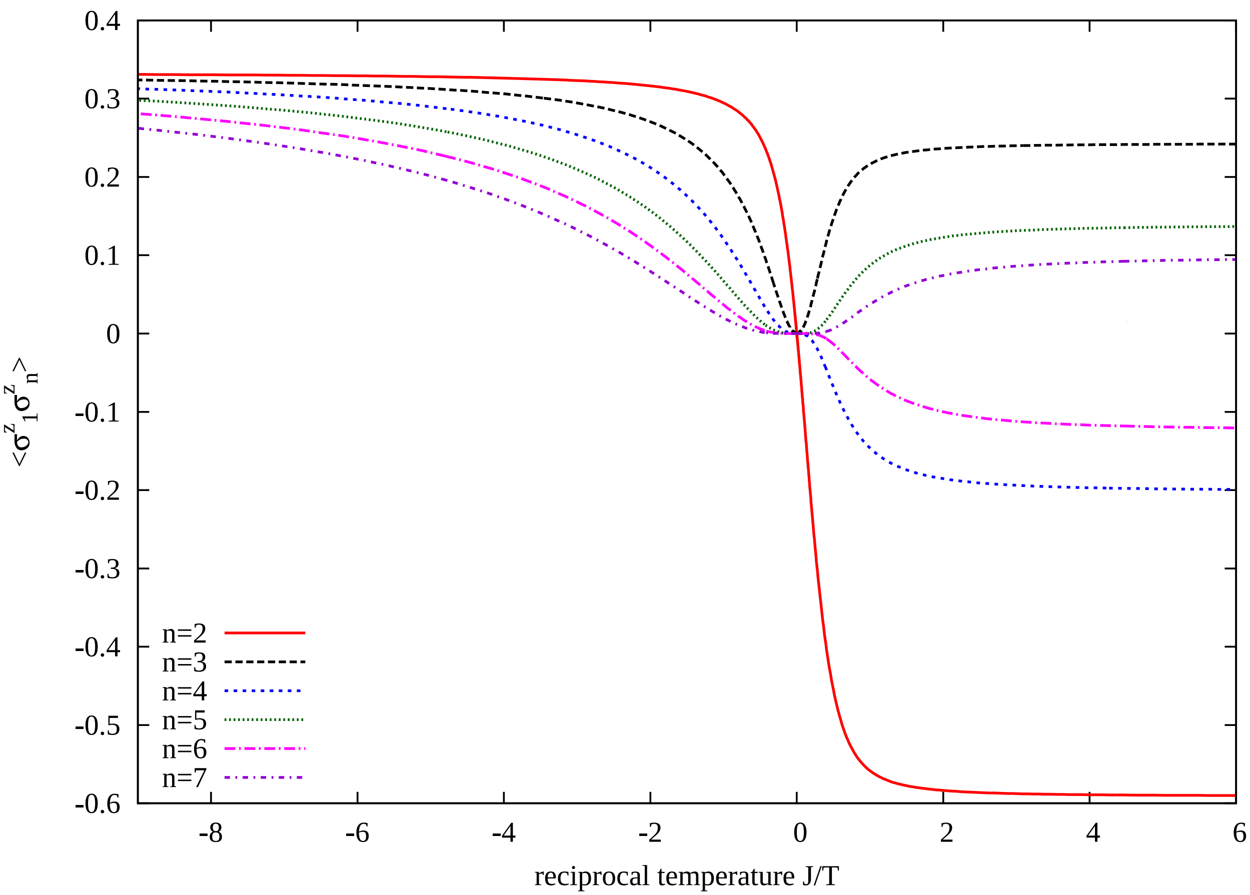
$$\omega(\lambda_1,\lambda_2) := \frac{1}{2} + \frac{(\lambda_1-\lambda_2)^2-1}{2\pi}\int_C \frac{d\mu}{1+a(\mu)}\,\frac{G(\mu,\mathrm{i}\lambda_1)}{(\mu-\mathrm{i}\lambda_2)(\mu-\mathrm{i}\lambda_2-\mathrm{i})}$$

where the functions G and a satisfy the linear and non-linear integral equations

$$G(\lambda,\lambda_1) = -\frac{1}{(\lambda-\lambda_1)(\lambda-\lambda_1-\mathrm{i})} + \frac{1}{\pi}\int_C \frac{d\mu}{1+a(\mu)}\,\frac{G(\mu,\lambda_1)}{1+(\lambda-\mu)^2}$$

$$\log a(\lambda) = 2J\frac{\beta}{\lambda(\lambda+\mathrm{i})} - \frac{1}{\pi}\int_C \frac{\log(1+a(\mu))}{1+(\lambda-\mu)^2}\,d\mu,$$

Sato, Aufgebauer, Boos, Göhmann, AK, Takahashi, Trippe (11)

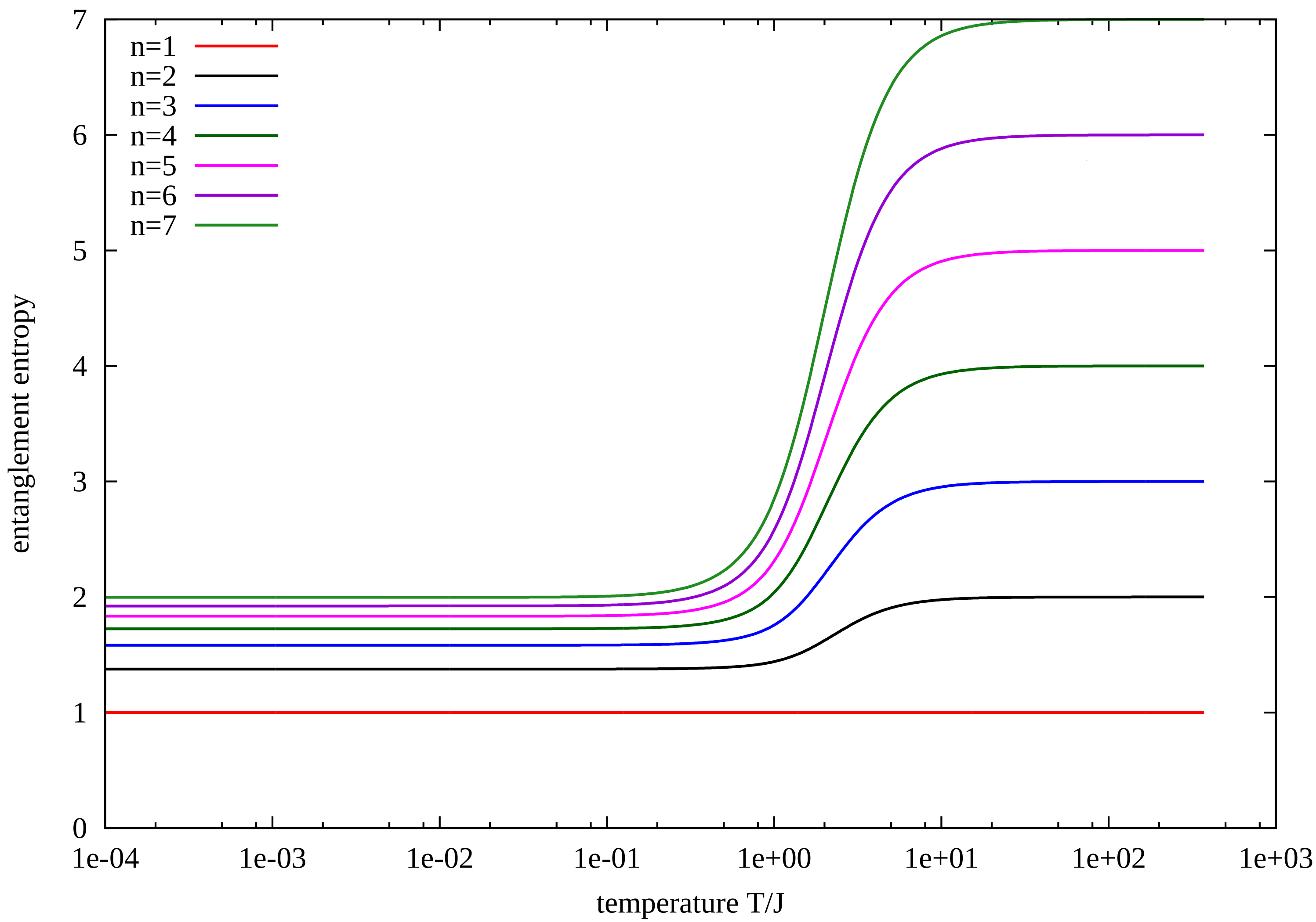


AF: low-temperature behaviour $\qquad \langle \sigma_1^z \sigma_{1+r}^z \rangle \simeq \langle \sigma_1^z \sigma_{1+r}^z \rangle_0 (1 - \gamma_r (T/J)^2)$

$$\langle \sigma_1^z \sigma_2^z \rangle \simeq \frac{1}{3} - \frac{4}{3} \ln 2 + \frac{1}{36} (T/J)^2, \qquad \langle \sigma_1^z \sigma_3^z \rangle \simeq \frac{1}{3} - \frac{16}{3} \ln 2 + 3\zeta(3) + \left(\frac{1}{9} - \frac{\pi^2}{72} \right) (T/J)^2$$

$su(2)$ $S = 1/2$: Finite temperatures – numerical results

AF case: entanglement entropy for n successive sites



Proofs and constructions

representation theory of quantum algebras/vertex operators ($T = h = 0$): Jimbo, Miki, Miwa, Nakayashiki 92; Jimbo, Miwa 95

functional equations, qKZ ($T = h = 0$): Jimbo, Miwa 96

algebraic Bethe ansatz: ($T = 0, h \neq 0$): Kitanine, Maillet, Terras 99
($T \geq 0, h \neq 0$): Göhmann, Klümper, Seel 04, 05; Boos, Göhmann 09

factorization of multiple integrals/correlation functions ($T = h = 0$): Boos, Korepin 01; Boos, Korepin, Smirnov 03, 04; Boos, Jimbo, Miwa, Smirnov, Takeyama 05, 06; Kato, Shiroishi, Takahashi, Sakai 03; Sato, Shiroishi, Takahashi 05

exponential formula for the reduced density matrix ($T = 0$): Boos, Jimbo, Miwa, Smirnov, Takeyama 06 (2 papers)

conjecture of exponential formula for finite temperature/finite length: Boos, Göhmann, Klümper, Suzuki 06; Damerau, Göhmann, Hasenclever, Klümper 07; Boos, Göhmann, Klümper, Suzuki 07

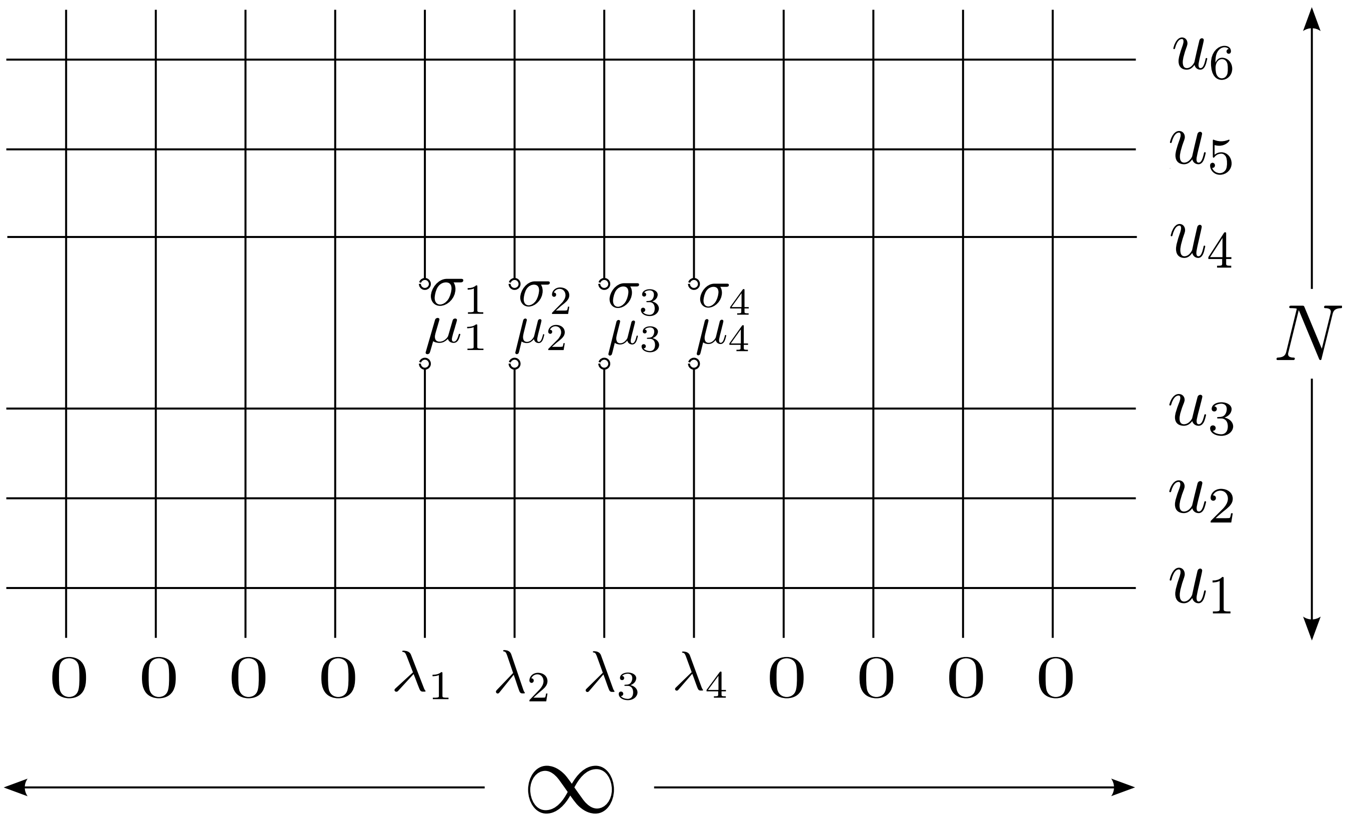
fermionic structure on space of local operators: Boos, Jimbo, Miwa, Smirnov, Takeyama 07, 09

algebraic proof of exponential formula/factorization: Jimbo, Miwa, Smirnov 09

algebraic/analytic proof: Aufgebauer, AK 12

Correlation functions/reduced density matrix

All correlation functions of a sequence of spin operators on consecutive n sites determined by reduced density matrix D_n with matrix elements $D_n{}^{(\mu_1,\mu_2,\ldots,\mu_n)}_{(\sigma_1,\sigma_2,\ldots,\sigma_n)}$. Finite Trotter number N.



$D_n(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is meromorphic: numerator = unknown n-variate polynomial of degree N

denominator = $\Lambda_0(\lambda_1)\cdot\ldots\cdot\Lambda_0(\lambda_n)$, largest eigenvals of QTM

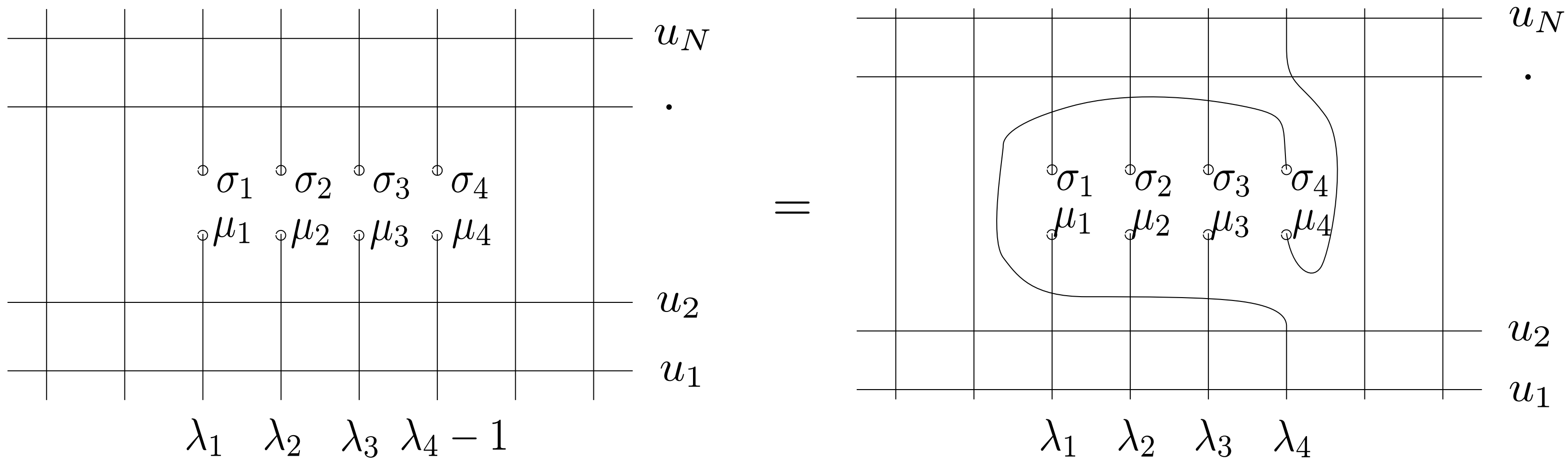
Discrete functional equation/unique solution

Functional equation ('rqKZ'-equation) for density matrix on $\infty \times N$–lattice

$$D_n(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - 1) = A(\lambda_1, \ldots, \lambda_n) D_n(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)$$

for arbitrary complex $\lambda_1, \ldots, \lambda_{n-1}$ and λ_n from set of spectral parameters on horizontal lines $\{u_1, u_2, \ldots, u_N\}$. (Sato et al. (11); Aufgebauer, AK (12))

A is a linear operator acting in the space of density matrices, example for $n = 4$:



Discrete functional equation + analyticity conditions + asymptotics fix D_n for arbitrary Trotter number N ($T \geq 0$). (Aufgebauer, AK (12))

$su(2)$ $S = 1/2$: The *mirror model* I

"Mirror model": Change of space and time direction

Change of driving term in NLIE (notice absence of Lorentz invariance)

$$2J\frac{\beta}{\lambda(\lambda+\mathrm{i})} \qquad \Longrightarrow \qquad L\log\frac{\lambda-\mathrm{i}/2}{\lambda+\mathrm{i}/2}$$

yielding

$$\omega(\lambda_1,\lambda_2) := \frac{1}{2} + \frac{(\lambda_1-\lambda_2)^2-1}{2\pi}\int_C \frac{d\mu}{1+a(\mu)}\frac{G(\mu,\mathrm{i}\lambda_1)}{(\mu-\mathrm{i}\lambda_2)(\mu-\mathrm{i}\lambda_2-\mathrm{i})}$$

where the functions G and a satisfy the linear and non-linear integral equations

$$G(\lambda,\lambda_1) = -\frac{1}{(\lambda-\lambda_1)(\lambda-\lambda_1-\mathrm{i})} + \frac{1}{\pi}\int_C \frac{d\mu}{1+a(\mu)}\frac{G(\mu,\lambda_1)}{1+(\lambda-\mu)^2}$$

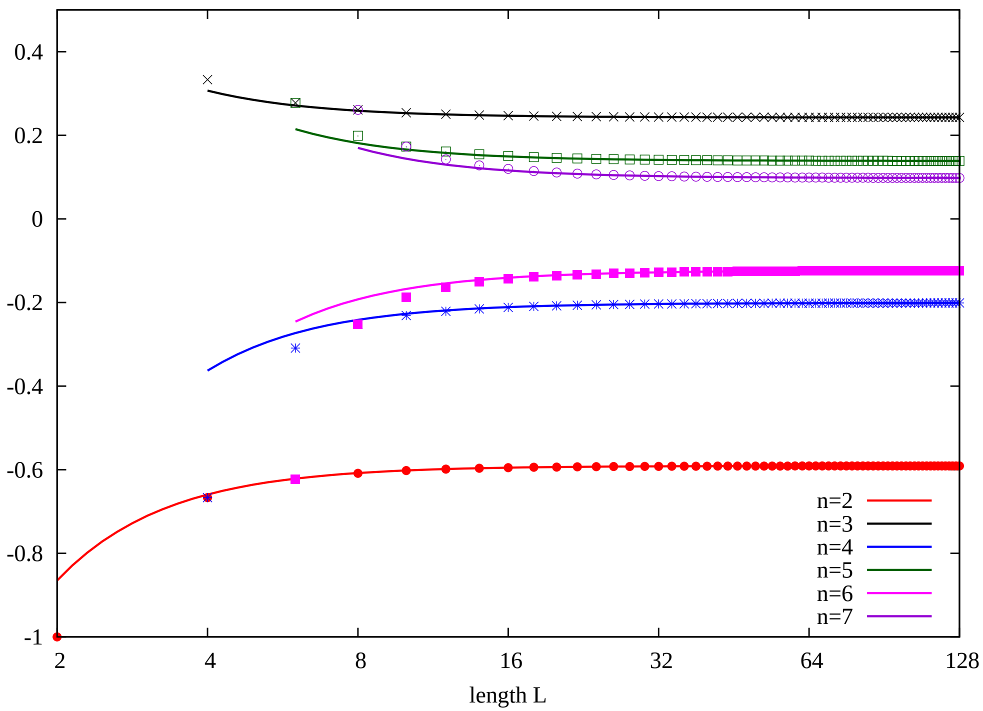
$$\log a(\lambda) = L\log\frac{\lambda-\mathrm{i}/2}{\lambda+\mathrm{i}/2} - \frac{1}{\pi}\int_C \frac{\log(1+a(\mu))}{1+(\lambda-\mu)^2}d\mu,$$

Sato, Aufgebauer, Boos, Göhmann, AK, Takahashi, Trippe (11)

AK, Batchelor (90); AK, Batchelor, Pearce (91); AK (92); Destri, de Vega (92)...

"Mirror model": Antiferromagnetic Heisenberg chain at $T = 0$ and $L = 2, 4, 6, \ldots$.

2-point correlators $\langle \sigma_1^z \sigma_n^z \rangle$ in dependence on chain length L for different point separations n.



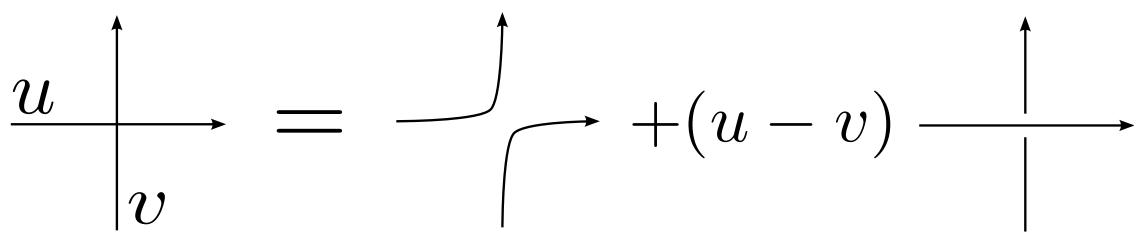
AF: finite size scaling $\langle \sigma_1^z \sigma_{1+r}^z \rangle \simeq \langle \sigma_1^z \sigma_{1+r}^z \rangle_0 (1 + 4\gamma_r \pi^2 / L^2)$.

Takhtajan-Babujian $S=1$ chain

$$H = \sum_j \left[\vec{S}_j \vec{S}_{j+1} - \left(\vec{S}_j \vec{S}_{j+1} \right)^2 \right]$$

Hamiltonians and conserved currents as log-derivatives of transfer matrices based on elementary spin-1/2 R-matrix

$$R^{(\frac{1}{2})(\frac{1}{2})}(u,v) =$$

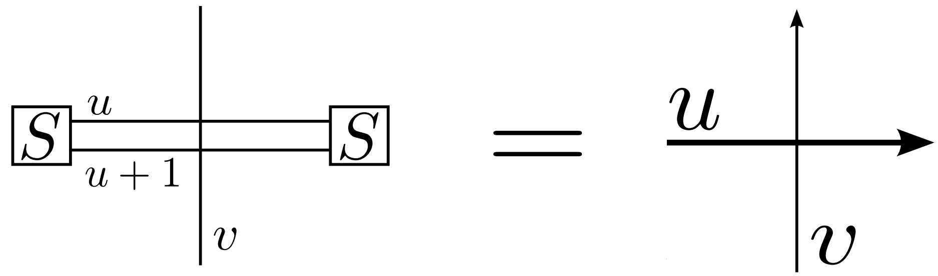


and fusion for spin-1 (and higher S)

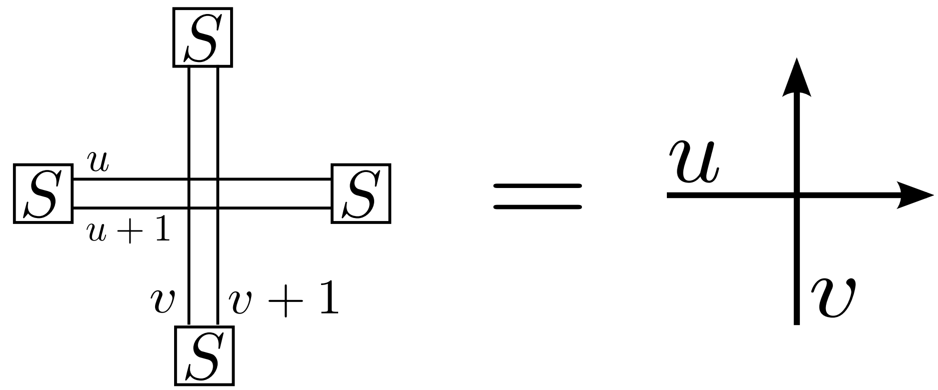
Fusion

Fusion for spin-1: mixed spin-1/2 $\times$ spin-1/2 and spin-1 $\times$ spin-1 (Kulish, Reshetikhin, Sklyanin 81)

$$R^{(1)(\frac{1}{2})}(u,v) =$$



$$R^{(1)(1)}(u,v) =$$



Factorization for spin-1/2 $\times$ spin-1/2 and spin-1 $\times$ spin-1, but not for mixed case (no functional equation)

Previous work

Idzumi (1993), Bougourzi, Weston (1994), Konno (1994):

higher spin XXX chains, multiple integral expressions, $T = 0$ by the vertex operator approach

N. Kitanine 2001:

higher spin XXX chains, multiple integral expressions, $T = 0$ by algebraic Bethe ansatz

T. Deguchi and C. Matsui 2010:

higher spin XXX and XXZ chains, $T = 0$, Bethe ansatz expressions

Göhmann, Seel, Suzuki 2010:

spin-1 XXX, multiple integral expressions, $T > 0$

Problem: simplest non-trivial case of nearest neighbour correlators leads to four-fold integral expressions (and difficult to evaluate)

Computation strategies for reduced density matrix

Two different strategies:

A) density matrix directly for (fully) fused case, i.e. all vertical lines carry spin-1
method: functional equations

B) calculation of density matrix for vertical spin-1/2 lines; then: fusion of pairs of spin-1/2 lines
method: factorization (note: functional equations do not exist)

Here: both A) D_2 for two spin-1

B) D_4 for four spin-1/2 (and general)

Generalization to D_n most systematically done along 2nd route.

Next page: functional equation for

$$D_2(\lambda_1,\lambda_2) = a(\lambda_1,\lambda_2)\cdot id + b(\lambda_1,\lambda_2)\cdot P + c(\lambda_1,\lambda_2)\cdot P_0$$

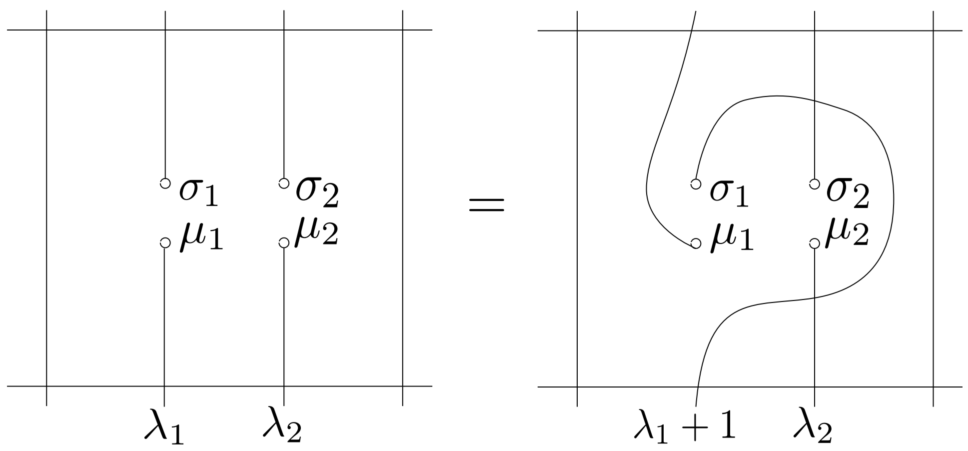
with permutation operator P and projector onto singlet P_0.

[Tr $D_2 = 1 = 9a + 3b + 3c$]

A: Functional equation

Functional equation for $\lambda_1 \in \{u_1, u_2, ..., u_N\}$ and arbitrary complex λ_2

$$D_2(\lambda_1,\lambda_2) = A(\lambda_1,\lambda_2) D_2(\lambda_1+1,\lambda_2)$$



Explicitly for $D_2(\lambda_1,\lambda_2) = a(\lambda_1,\lambda_2)\cdot id + b(\lambda_1,\lambda_2)\cdot P + c(\lambda_1,\lambda_2)\cdot P_0$ $\qquad (\lambda := \lambda_1 - \lambda_2)$

$$\begin{bmatrix} a(\lambda_1,\lambda_2) \\ b(\lambda_1,\lambda_2) \\ c(\lambda_1,\lambda_2) \end{bmatrix} = \begin{bmatrix} \frac{-6\lambda-4+\lambda^2+4\lambda^3+\lambda^4}{\lambda(\lambda+2)(\lambda+3)(\lambda-1)} & \frac{2(-2-2\lambda+\lambda^2+\lambda^3)}{\lambda(\lambda+2)(\lambda+3)(\lambda-1)} & \frac{2(\lambda+1)}{(\lambda+3)(\lambda+2)} \\ \frac{4(2\lambda+1+\lambda^2)}{\lambda(\lambda+2)(\lambda+3)(\lambda-1)} & \frac{2(4+5\lambda-\lambda^3)}{\lambda(\lambda+2)(\lambda+3)(\lambda-1)} & \frac{(\lambda+1)\lambda}{(\lambda+3)(\lambda+2)} \\ \frac{-4(2\lambda-2+\lambda^2)}{\lambda(\lambda+2)(\lambda+3)(\lambda-1)} & \frac{4-4\lambda-5\lambda^2+\lambda^4}{\lambda(\lambda+2)(\lambda+3)(\lambda-1)} & \frac{-2\lambda}{(\lambda+3)(\lambda+2)} \end{bmatrix} \cdot \begin{bmatrix} a(\lambda_1+1,\lambda_2) \\ b(\lambda_1+1,\lambda_2) \\ c(\lambda_1+1,\lambda_2) \end{bmatrix}$$

A: Results for $D_2^{(1)}$

Explicit results

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{9} - \frac{2}{15\lambda^2(\lambda^2-1)} & -\frac{1}{30}\lambda^2(\lambda^2-4) & \frac{1}{30}(2\lambda^2-3)(\lambda^2-1)(\lambda^2-4) \\ \frac{2(5\lambda^2+7)}{45\lambda^2(\lambda^2-1)} & \frac{1}{60}(5+3\lambda^2)(\lambda^2-4) & -\frac{1}{30}(3\lambda^2-7)(\lambda^2-1)(\lambda^2-4) \\ -\frac{2(5\lambda^2-2)}{45\lambda^2(\lambda^2-1)} & \frac{1}{60}(3\lambda^2-5)(\lambda^2-4) & -\frac{1}{30}(3\lambda^2-2)(\lambda^2-1)(\lambda^2-4) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ g \\ h \end{bmatrix}$$

where a, b, c, g, h are functions of λ_1, λ_2 and $\lambda := \lambda_1 - \lambda_2$

$$g = \frac{4}{3N}\left(\Omega(\lambda_1,\lambda_2) + \Omega(\lambda_1+1,\lambda_2+1) + \Omega(\lambda_1,\lambda_2+1) + \Omega(\lambda_1+1,\lambda_2) - \frac{2}{\lambda^2-4}\right)$$

$$h = \frac{4}{3N}\left(\Omega(\lambda_1,\lambda_2)\,\Omega(\lambda_1+1,\lambda_2+1) - \Omega(\lambda_1,\lambda_2+1)\,\Omega(\lambda_1+1,\lambda_2) - \frac{(\lambda^4+\lambda^2+1)}{(\lambda^2-4)(\lambda^2-1)^2\lambda^2}\right.$$
$$\left. + \frac{(\lambda^2-\lambda+1)}{(\lambda^2-1)\lambda(\lambda-2)}\Omega(\lambda_1+1,\lambda_2) + \frac{(\lambda^2+\lambda+1)}{(\lambda^2-1)\lambda(\lambda+2)}\Omega(\lambda_1,\lambda_2+1)\right)$$

$$N = \left(\frac{3}{2} + \omega(\lambda_1,\lambda_1+1)\right)\left(\frac{3}{2} + \omega(\lambda_2,\lambda_2+1)\right), \qquad \Omega(\lambda_1,\lambda_2) = \frac{\omega(\lambda_1,\lambda_2) + \frac{1}{2}}{\lambda^2-1}$$

A: Elementary function $\omega(\lambda_1, \lambda_2)$

Non-linear integral equations (ad hoc: AK, Batchelor, Pearce 91; systematic: Suzuki 98/99)

$$\begin{pmatrix} \ln y \\ \ln a \\ \ln \bar{a} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\frac{\pi}{2}\beta}{\cosh\frac{\pi}{2}x} \\ -\frac{\frac{\pi}{2}\beta}{\cosh\frac{\pi}{2}x} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \kappa(x) & \kappa(x) \\ \kappa(x) & F(x) & -F(x+2i) \\ \kappa(x) & -F(x-2i) & F(x) \end{pmatrix}}_{=:K(x)} * \begin{pmatrix} \ln(1+y) \\ \ln(1+a) \\ \ln(1+\bar{a}) \end{pmatrix}$$

where $\kappa(x) \quad := \quad \frac{1}{4\cosh\frac{\pi}{2}x}, \qquad F(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-|q|}}{2\cosh q} e^{-iqx} dq$

New results (Aufgebauer, Göhmann, AK, Nawrath, Suzuki 12):

$$\Omega(\lambda_1, \lambda_2) = \frac{1}{(\lambda_1-\lambda_2)^2-1} + \int_{-\infty}^{\infty} \frac{\pi i}{\cosh\frac{\pi}{2}(2i\lambda_1 - x)} \left[\frac{4}{(x-2i\lambda_2)^2+1} + \frac{1}{2}Y(x)\right] dx$$

$$\begin{pmatrix} (1+1/y)Y \\ (1+1/a)A \\ (1+1/\bar{a})\bar{A} \end{pmatrix} = \begin{pmatrix} \frac{\pi}{\cosh\frac{\pi}{2}(x-2i\lambda_2)} \\ 0 \\ 0 \end{pmatrix} + K * \begin{pmatrix} Y \\ A \\ \bar{A} \end{pmatrix}$$

A: Solution procedure: Analyticity conditions

Further conditions on $D_2(\lambda_1,\lambda_2)$:

1) matrix elements of $\Lambda(\lambda_1)\Lambda(\lambda_2)D_2(\lambda_1,\lambda_2)$ are polynomials of degree $2N$, where Λ is the leading eigenvalue of the (vertical) spin-1 transfer matrix.

2) $\lim_{\lambda_2\to\infty} D_2(\lambda_1,\lambda_2) = D_1(\lambda_1)\otimes$ id $= \frac{1}{3}$ id

3) intertwining $D_2(\lambda_1,\lambda_2) = R(\lambda_1,\lambda_2)D_2(\lambda_1,\lambda_2)R^{-1}(\lambda_1,\lambda_2) = D_2(\lambda_2,\lambda_1)$

Note:

The $N+1$ many conditions functional equations and asymptotics 2) for the $2N+1$ many coefficients appearing in 1) are underdetermined.

Together with 3) we have a fully determined system of equations for the coefficients.

The solution can be written algebraically in terms of a rational function $\omega(\lambda_1,\lambda_2)$ satisfying

$$\frac{1}{N(\lambda_1,\lambda_2)}\left(\Omega(\lambda_1,\lambda_2)+\Omega(\lambda_1+1,\lambda_2)-\frac{\lambda^2+\lambda+1}{(\lambda+2)\lambda\left(-1+\lambda^2\right)}\right)+(\lambda_1\to\lambda_1+1)=0$$

for discrete values of λ_1, where $N(\lambda_1,\lambda_2)$ and $\Omega(\lambda_1,\lambda_2)$ are related to $\omega(\lambda_1,\lambda_2)$ as before.

B: Similarities to factorization formulas of spin-1/2 Heisenberg

After having done the calculations, close similarities to the algebraic structures of the spin-1/2 Heisenberg chain became obvious.

Let us look at the $D_4^{(\frac{1}{2})}$ density matrix: four vertical spin-1/2 lines in "sea" of spin-1 lines.

Take the known formulas for $D_4^{(\frac{1}{2})}$ (Sato, Shiroishi, Takahashi 05/06) with $\omega(\lambda_1,\lambda_2)$ obtained from $D_2^{(\frac{1}{2})} = (\frac{1}{4} - \frac{1}{6}\omega)\mathrm{id} + \frac{1}{3}\omega P$ and calculate $D_4^{(\frac{1}{2})}{}_{++++}^{++++}(\xi_1,\xi_2,\xi_3,\xi_4)$ and $D_4^{(\frac{1}{2})}{}_{++--}^{++--}(\xi_1,\xi_2,\xi_3,\xi_4)$.

Then set $\xi_1 = \lambda_1$, $\xi_2 = \lambda_1 + 1$, $\xi_3 = \lambda_2$, $\xi_4 = \lambda_2 + 1$ from which

$$
\begin{aligned}
D_4^{(\frac{1}{2})}{}_{++++}^{++++}(\lambda_1,\lambda_1+1,\lambda_2,\lambda_2+1) &\quad\to\quad D_2^{(1)}{}_{+1+1}^{+1+1}(\lambda_1,\lambda_2), \\
D_4^{(\frac{1}{2})}{}_{++--}^{++--}(\lambda_1,\lambda_1+1,\lambda_2,\lambda_2+1) &\quad\to\quad D_2^{(1)}{}_{+1-1}^{+1-1}(\lambda_1,\lambda_2),
\end{aligned}
$$

after a suitable normalization.

From the two matrix elements, $a(\lambda_1,\lambda_2)$, $b(\lambda_1,\lambda_2)$, and $c(\lambda_1,\lambda_2)$ can be calculated...

...and agree with the formulas presented above!

Factorization seems to hold for the spin-1/2 density matrix $D_n^{(\frac{1}{2})}$.

B: Proof of factorization for $D_m^{(\frac{1}{2})}$ (I)

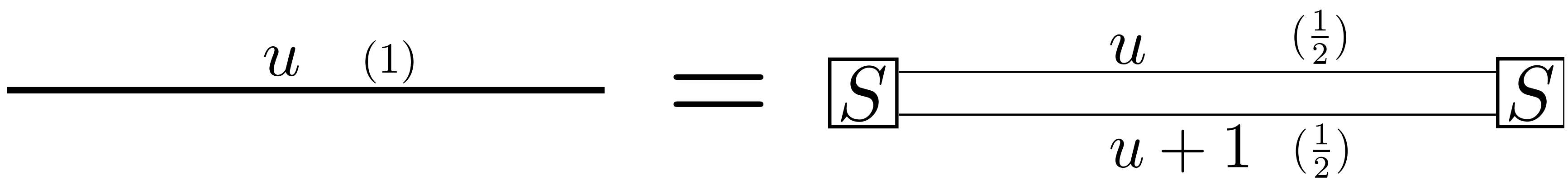
(positive) test by computer algebra:

$N = 2$ horizontal spin-1 lines & four vertical spin-1/2 lines, infinitely many vertical spin-1 lines

Proof?

Usually done by functional equations, but do not exist for the mixed spin R matrices.

First idea: take each horizontal spin-1 line as fusion of two spin-1/2 lines



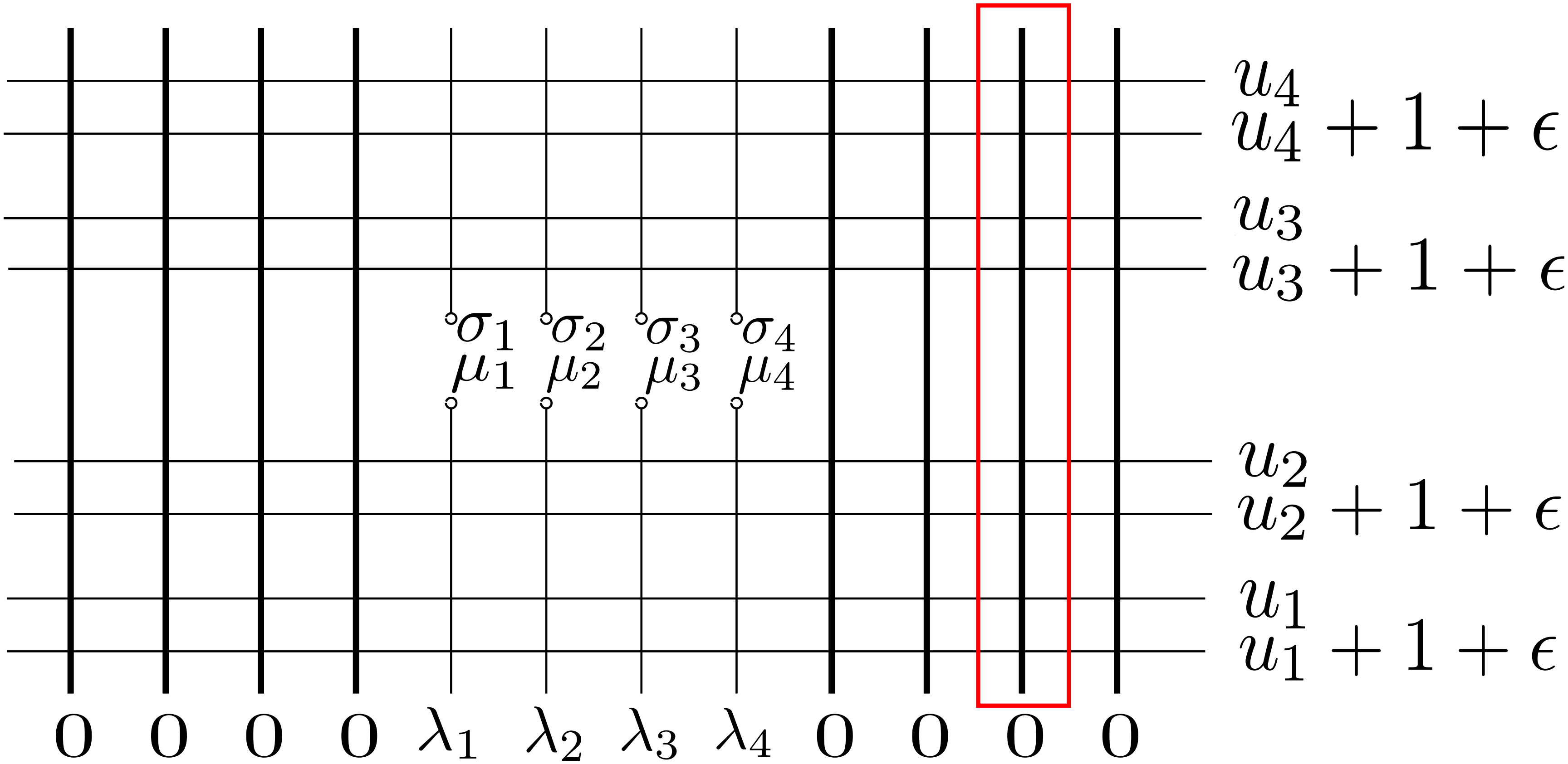
Problems:

- How to get rid of the symmetrization operators?
- Simultaneous occurrence of u and $u+1$ makes derivation of functional equation for $\lambda_1 = u$ and $u+1$ impossible ($0/0$ expressions)

Also: $\left(\Omega(\lambda_1, \lambda_2) + \Omega(\lambda_1 + 1, \lambda_2) - \frac{\lambda^2+\lambda+1}{(\lambda+2)\lambda(-1+\lambda^2)}\right)$ is nonzero, but satisfies a functional equation itself.

B: Proof of factorization for $D_n^{(\frac{1}{2})}$ (II)

We deal with regularized system



leading eigenstate of column-to-column transfer matrix converges for $\varepsilon \to 0$ to state in tensor product of local spin-1 spaces (details lengthy).

B: Proof of factorization for $D_n^{(\frac{1}{2})}$ (III)

Further line of reasoning

- Functional equations hold for $\varepsilon > 0$.
- Factorized expressions for density matrix holds for $\varepsilon > 0$.
- Limit $\varepsilon \to 0$ is non-singular for generic values of λ_1....

“Self fusion”

Summary

Presentation of algebraic expressions for correlation functions of the integrable spin-1/2 and spin-1 XXX chain at arbitrary temperature (system size)

- Functional equations approach
- Factorized expressions for density matrix on basis of “old” spin-1/2 expressions

Open problems: arbitrary spin-S, XXZ and finite magnetic field – probably simple

Solution procedure: Functional equation

Simplification after transformation

$$\begin{bmatrix} f(\lambda_1,\lambda_2) \\ g(\lambda_1,\lambda_2) \\ h(\lambda_1,\lambda_2) \end{bmatrix} := \begin{bmatrix} 9 & 3 & 3 \\ \frac{-16}{(\lambda^2-4)(\lambda^2-1)} & \frac{4(\lambda^2-2)}{(\lambda^2-4)(\lambda^2-1)} & \frac{-4\left(\lambda^2-1\right)}{(\lambda^2-4)(\lambda^2-1)} \\ -\frac{4(2\lambda^2+3)}{(\lambda^2-4)(\lambda^2-1)^2\lambda^2} & \frac{2(\lambda^4-3\lambda^2-2)}{(\lambda^2-4)(\lambda^2-1)^2\lambda^2} & \frac{2(\lambda^2+2)(\lambda^2+1)}{(\lambda^2-4)(\lambda^2-1)^2\lambda^2} \end{bmatrix} \cdot \begin{bmatrix} a(\lambda_1,\lambda_2) \\ b(\lambda_1,\lambda_2) \\ c(\lambda_1,\lambda_2) \end{bmatrix}$$

to triangular form

$$\begin{bmatrix} f(\lambda_1,\lambda_2) \\ g(\lambda_1,\lambda_2) \\ h(\lambda_1,\lambda_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{\lambda^2+\lambda+1}{(\lambda+2)\lambda(\lambda^2-1)} & 1 \end{bmatrix} \cdot \begin{bmatrix} f(\lambda_1+1,\lambda_2) \\ g(\lambda_1+1,\lambda_2) \\ h(\lambda_1+1,\lambda_2) \end{bmatrix}$$

Results for $D_2^{(1)}$: alternative formulation

Explicit results

$$\begin{bmatrix} a(\lambda_1,\lambda_2) \\ b(\lambda_1,\lambda_2) \\ c(\lambda_1,\lambda_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{9} - \frac{2}{15}\,\frac{1}{\lambda^2(\lambda^2-1)} & -\frac{1}{30}\,\frac{\lambda^2}{\lambda^2-1} & \frac{1}{30}\,\frac{2\lambda^2-3}{\lambda^2-1} \\ \frac{2}{45}\,\frac{5\lambda^2+7}{\lambda^2(\lambda^2-1)} & \frac{1}{60}\,\frac{3\lambda^2+5}{\lambda^2-1} & -\frac{1}{30}\,\frac{3\lambda^2-7}{\lambda^2-1} \\ -\frac{2}{45}\,\frac{5\lambda^2-2}{\lambda^2(\lambda^2-1)} & \frac{1}{60}\,\frac{3\lambda^2-5}{\lambda^2-1} & -\frac{1}{30}\,\frac{3\lambda^2-2}{\lambda^2-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ G(\lambda_1,\lambda_2) \\ H(\lambda_1,\lambda_2) \end{bmatrix}, \qquad \lambda := \lambda_1 - \lambda_2$$

$$G = \frac{4}{3N}\Big((\lambda^2-4)\left[\Omega(\lambda_1,\lambda_2)+\Omega(\lambda_1+1,\lambda_2+1)+\Omega(\lambda_1,\lambda_2+1)+\Omega(\lambda_1+1,\lambda_2)\right]-2\Big)(\lambda^2-1$$

$$H = \frac{4}{3N}\Bigg((\lambda^2-4)(\lambda^2-1)^2\Big[\Omega(\lambda_1,\lambda_2)\,\Omega(\lambda_1+1,\lambda_2+1)-\Omega(\lambda_1,\lambda_2+1)\,\Omega(\lambda_1+1,\lambda_2)$$

$$+\ \frac{(\lambda^2-\lambda+1)}{(\lambda^2-1)\,\lambda\,(\lambda-2)}\Omega(\lambda_1+1,\lambda_2)+\frac{(\lambda^2+\lambda+1)}{(\lambda^2-1)\,\lambda\,(\lambda+2)}\Omega(\lambda_1,\lambda_2+1)\Big]-\frac{(\lambda^4+\lambda^2+1)}{\lambda^2}\Bigg)$$

$$N = \left(\frac{3}{2}+\omega(\lambda_1,\lambda_1+1)\right)\left(\frac{3}{2}+\omega(\lambda_2,\lambda_2+1)\right), \qquad \Omega(\lambda_1,\lambda_2) = \frac{\omega(\lambda_1,\lambda_2)+\frac{1}{2}}{\lambda^2-1}$$

Functional equation II: alternative formulation

Simplification after transformation

$$\begin{bmatrix} F(\lambda_1,\lambda_2) \\ G(\lambda_1,\lambda_2) \\ H(\lambda_1,\lambda_2) \end{bmatrix} := \begin{bmatrix} 9 & 3 & 3 \\ -16 & 4\lambda^2-8 & -4\lambda^2+4 \\ -8-\frac{12}{\lambda^2} & 2\lambda^2-6-\frac{4}{\lambda^2} & -2\lambda^2-6-\frac{4}{\lambda^2} \end{bmatrix} \cdot \begin{bmatrix} a(\lambda_1,\lambda_2) \\ b(\lambda_1,\lambda_2) \\ c(\lambda_1,\lambda_2) \end{bmatrix}$$

to triangular form

$$\begin{bmatrix} F(\lambda_1,\lambda_2) \\ G(\lambda_1,\lambda_2) \\ H(\lambda_1,\lambda_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{(\lambda-2)(\lambda+1)}{\lambda(\lambda+3)} & 0 \\ 0 & -\frac{(\lambda-2)(\lambda+1)\left(\lambda^2+\lambda+1\right)}{\lambda^2(\lambda+3)(\lambda+2)} & \frac{(\lambda-1)(\lambda-2)(\lambda+1)^2}{\lambda^2(\lambda+3)(\lambda+2)} \end{bmatrix} \cdot \begin{bmatrix} F(\lambda_1+1,\lambda_2) \\ G(\lambda_1+1,\lambda_2) \\ H(\lambda_1+1,\lambda_2) \end{bmatrix}$$