Higher Grading Structure of the WKI Hierarchy and the Short Pulse Equation [1]

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Aim

Discuss the algebraic structure of the *WKI Hierarchy* containing equations like:

- the *short pulse equation*,

$$u_{xt} = 4u + \frac{2}{3}(u^3)_{xx}$$

- the *elastic beam equation*,

$$u_{xt} = \frac{1}{4}\frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

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The Higher Grading Structure

- Consider an Affine Algebra $\hat{\mathcal{G}}$ and a grading operator Q such that,

$$\hat{\mathcal{G}} = \oplus \mathcal{G}_n, \qquad n \in Z$$

where

$$[Q, \mathcal{G}_n] = n\mathcal{G}_n, \qquad [\mathcal{G}_m, \mathcal{G}_n] \in \mathcal{G}_{m+n}$$

- Consider the zero curvature representation

$$[\partial_x + U, \;\; \partial_{t_{mn}} + V\,] = 0$$

$$U = E^{(1)} + A^{(1)}[\phi], \qquad V = \sum_{i=-m}^{n} D^{(i)}[\phi]$$

where $E^{(1)} \in \mathcal{K}^{(1)} \in Ker_E$ and $A^{(1)} \in \mathcal{M}^{(1)}$ have *both grade one*, where $\mathcal{G}_j = \mathcal{K}^{(j)} \oplus \mathcal{M}^{(j)}$ and

$$\mathcal{K}^{(j)} = \{x \in \mathcal{G}_j, [x, E^{(1)}] = 0\}.$$

In particular for the WKI *hierarchy*

$$\hat{\mathcal{G}} = \hat{sl}(2), \quad \mathcal{G}_j = \{\ h^{(j)},\ E_{\pm\alpha}^{(j)}\ \},$$

$$Q = d, \qquad E^{(1)} = h^{(1)}$$

and

$$A^{(1)}[q,r] = qE_{-\alpha}^{(1)} + rE_{\alpha}^{(1)}$$

- **Positive Flows**

$$[\partial_x + E^{(1)} + A^{(1)}, \partial_{t_n} + D^{(n)} + \cdots D^{(1)}] = 0$$

Decompose according to grade and solve for $D^{(j)}$,i.e

$$[E^{(1)} + A^{(1)}, D^{(n)}] = 0,$$
$$\partial_x D^{(n)} + [E^{(1)} + A^{(1)}, D^{(n-1)}] = 0,$$
$$\vdots$$
$$\partial_x D^{(1)} - \partial_{t_n} A^{(1)} + [E^{(1)} + A^{(1)}, D^{(0)}] = 0,$$

- For $n = 2$
  get *Wadati-Konno-Ichikawa-Shimizu* eqn.

$$q_{t_2} = -(\frac{q}{\sqrt{1+qr}})_{xx}, \quad r_{t_2} = (\frac{r}{\sqrt{1+qr}})_{xx}$$

- For $n = 3$
  get *Generalized Elastic Beam* eqn.

$$q_{t_3} = -(\frac{q_x}{(1+qr)^{3/2}})_{xx}, \quad r_{t_3} = (\frac{r_x}{(1+qr)^{3/2}})_{xx}$$

For $r = 1, \quad q = v - 1$ get *Dym eqn*,

$$v_t = (v^{-1/2})_{xxx}$$

- **Negative Flows**

  $$[\partial_x + E^{(1)} + A^{(1)}, \partial_{t_{-n}} + D^{(-n)} + \cdots D^{(1)}] = 0$$

- For $n = -1$ get

  $$\begin{aligned} \partial_{t_{-1}} q &= 4\partial^{-1} q + 2\partial \left[q\partial^{-1} \left(r\partial^{-1} q + q\partial^{-1} r \right) \right], \\ \partial_{t_{-1}} r &= 4\partial^{-1} r + 2\partial \left[r\partial^{-1} \left(r\partial^{-1} q + q\partial^{-1} r \right) \right]. \end{aligned}$$

  Introducing new fields

  $$q \equiv \partial_x u, \qquad r \equiv \partial_x v, \tag{1}$$

  we can write in a local form yielding the *generalized two-component short pulse* eqn.

  $$\begin{aligned} u_{xt_{-1}} &= 4u + 2\partial_x \left(uvu_x \right), \\ v_{xt_{-1}} &= 4v + 2\partial_x \left(uvv_x \right). \end{aligned}$$

- **Soliton Solutions**

The idea of the dressing method is to transform the vacuum into a non trivial solution by gauge transformation, i.e.,

$$E^{(1)} + A_0 = \Theta_\pm E^{(1)} \Theta_\pm^{-1} - \partial_x \Theta_\pm \Theta_\pm^{-1},$$

where $A_0 \in \mathcal{M}^{(0)}$ and

$$\Theta_+ = e^{X^{(0)}} e^{X^{(1)}} \cdots, \quad \Theta_- = e^{Y^{(0)}} e^{Y^{(-1)}} \cdots$$

Since for the WKI hierarchy, the fields lie in $A^{(1)} = qE_\alpha^{(1)} + rE_{-\alpha}^{(1)} \in \mathcal{M}^{(1)}$ instead of A_0, we find,

$$qE_\alpha^{(1)} + rE_{-\alpha}^{(1)} = -\partial_x (e^{X(1)}) e^{-X(1)}$$

and

$$H^{(1)} + qE_\alpha^{(1)} + rE_{-\alpha}^{(1)} = e^{Y(0)} H^{(1)} e^{-Y(0)},$$

Since $Y(0) \in \mathcal{G}_0$ it can be parametrized as

$$e^{Y(0)} = e^{\chi E_{-\alpha}^{(0)}} \; e^{\phi H^{(0)}} \; e^{\psi E_\alpha^{(0)}}$$

we then obtain an inconsistency

$$H^{(1)} + qE^{(1)}_{\alpha} + rE^{(1)}_{-\alpha} = \left(1 + 2\chi\psi e^{2\phi}\right) H^{(1)}$$

$$-\ 2\psi e^{2\phi} E^{(1)}_{\alpha} + \left(2\chi + 2\psi\chi^2 e^{2\phi}\right) E^{(1)}_{-\alpha}.$$

In order to eliminate the inconsistency above (coef. of $H^{(1)}$) we propose a change of coordinates (*reciprocal coordinates*), $(y, s) \to (x, t)$ as follows

$$dx = \alpha dy - \beta ds, \qquad dt = ds$$

where α and β satisfy the continuity eqn.

$$\partial_s \alpha = -\partial_y \beta.$$

In such case, let $f(x, t) = f(x(y, s), t(y, s))$,

$$\alpha \partial_x f = \partial_y f,$$

Consider the AKNS Hierarchy on (y, s) coordinates,

$$\Big(\partial_y + F\Big)\,\Phi = 0, \quad (\partial_s + G)\,\Phi = 0$$

where $F = H^{(1)} + WE_{\alpha}^{(0)} + ZE_{-\alpha}^{(0)}$. Under gauge transformation

$$F \rightarrow \tilde{F} = B^{-1}FB - \partial_y B^{-1}B,$$

with B satisfying

$$WE_{\alpha}^{(0)} + ZE_{-\alpha}^{(0)} = B\partial_y B^{-1}$$

we find,

$$\begin{aligned}
\tilde{F} &= B^{-1}(F - B\partial_y B^{-1})B = B^{-1}H^{(1)}B \\
&= (1 + \psi\chi e^{2\phi})H^{(1)} - 2\chi e^{2\phi}E_{-\alpha}^{(1)} \\
&+ (2\psi + 2\psi\chi^2 e^{2\phi})E_{\alpha}^{(1)}
\end{aligned}$$

for $B = e^{\chi E_{-\alpha}}\, e^{\phi H}\, e^{\psi E_{\alpha}}$.

From the reciprocal changing of coordinates ,

$$(\partial_y + F)\Phi = 0 \rightarrow \alpha(\partial_x + \frac{\tilde{F}(x,t)}{\alpha})\tilde{\Phi} = 0$$

we can choose $\alpha = 1 + 2\psi\chi e^{2\phi}$ such that $\tilde{F}$ has the WKI form, i.e.,

$$\tilde{F} = H^{(1)} + qE^{(1)}_{-\alpha} + rE^{(1)}_{\alpha}$$

where

$$r = \frac{2\chi e^{2\phi}}{\alpha}, \qquad q = 2\frac{(1 + \chi\psi^2 e^{2\phi})}{\alpha}$$

It therefore follows that the AKNS hierarchy in coordinates (y, s) is mapped into the WKI hirarchy with coodinates (x, t) where,

$$\begin{aligned} x(y,t) &= \int \alpha(y,t)dy + const, \\ &= y + 2\int \chi\psi e^{2\phi}dy + const \end{aligned}$$