Higher Spin 6-Vertex Model and Macdonald Polynomials

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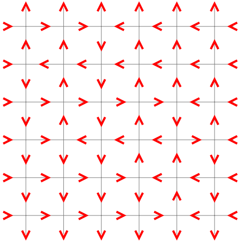
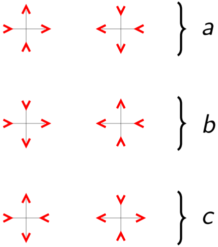
Joint work with Ferenc Balogh

Outline

Definition of the 6-Vertex model





Definition (Partition function)

The partition function is defined as usual:

$$\mathcal{Z}_n \propto \sum_{\text{configurations}} \prod_{i,j} \omega_{i,j}$$

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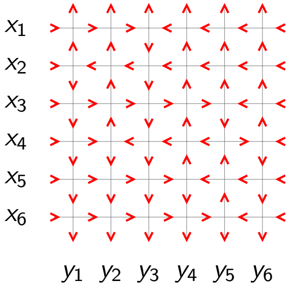

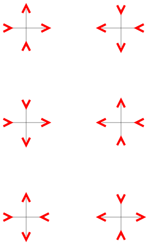

$$a = qx - y$$

$$b = x - qy$$

$$c = (q^{-1} - q)\sqrt{qxy}$$

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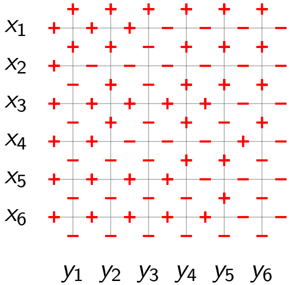

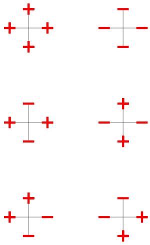

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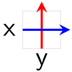
The $\check{R}$ matrix

We can build a configuration using boxes:



$$= \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} =: \check{R}(x, y)$$

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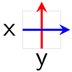

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Then

$$\check{R}(x,y) : V_1 \otimes V_2 \to V_2 \otimes V_1$$

where V_i is the spin 2 dimensional representation of $U_q(sl_2)$.

Yang–Baxter equation

Identity equation:

$$\check{R}(x,y)\check{R}(y,x) = (qx-y)(qy-x)Id.$$

Yang–Baxter equation:

$$\check{R}_{2,3}(y_2,y_3)\check{R}_{1,3}(y_1,y_3)\check{R}_{1,2}(y_1,y_2) = \check{R}_{1,2}(y_1,y_2)\check{R}_{1,3}(y_1,y_3)\check{R}_{2,3}(y_2,y_3)$$

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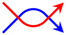 $= (qx - y)(qy - x)$ 

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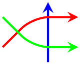
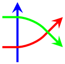
 $=$ 

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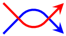
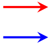
 $= (qx - y)(qy - x)$ 

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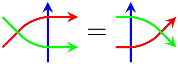
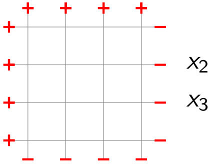


Using Yang–Baxter equation, we can show that $\mathcal{Z}_n$ is symmetric in x:



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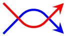 $= (qx - y)(qy - x)$ 

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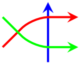
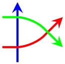
 $=$ 

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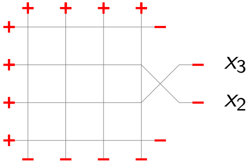
$\frac{1}{qx_3 - x_2}$ 

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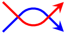 $= (qx - y)(qy - x)$ 

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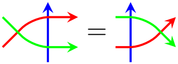

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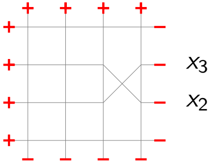
$\frac{1}{qx_3 - x_2}$ 

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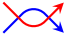
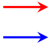
 $= (qx - y)(qy - x)$ 

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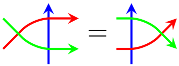

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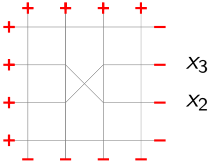
$\frac{1}{qx_3 - x_2}$ 

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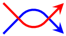 $= (qx - y)(qy - x)$ 

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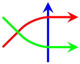
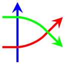
 $=$ 

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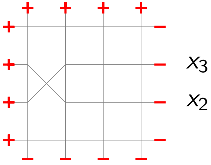

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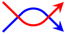 $= (qx - y)(qy - x)$ 

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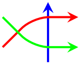
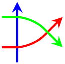
 $=$ 

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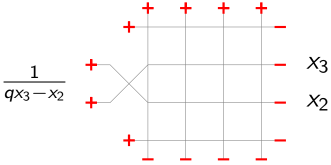

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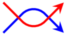
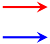
 $= (qx - y)(qy - x)$ 

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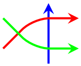
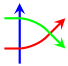
 $=$ 

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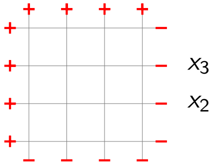

Explicit formula for the partition function

Using Korepin's recursion relation, Izergin showed that:

$$\mathcal{Z}_n(x, y) = (\text{Pre-factor}) \det \left| \frac{1}{(x_i - q y_j)(q x_i - y_j)} \right|_{i,j}$$

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Properties

- *Satisfies the Korepin's recursion relation;*
- *A homogeneous polynomial;*
- *Has total degree $n(n-1)$;*
- *Is symmetric in x, and y.*

A Schur polynomial

Set $q = e^{\frac{2i\pi}{3}}$, the so-called combinatorial point.

$$Y_n = \{n-1, n-1, n-2, n-2, \ldots, 1, 1, 0, 0\}$$

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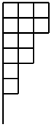
$Y_4 =$ 

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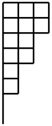

It turns out that:

$$\mathcal{Z}_n(x, y) = S_{Y_n}(x, y)$$

Definition (Schur polynomial)

Let $\lambda = \{\lambda_1, \dots, \lambda_n\}$, then the Schur polynomial is given by:

$$S_\lambda(x) = \frac{\det \left| x_i^{n+\lambda_j - j} \right|_{i,j}}{\det \left| x_i^{n-j} \right|_{i,j}}$$

Higher Spin Generalization

We can build a higher spin version, of the model:

$$\check{R}^{(\ell)}(x,y) : V_1^{(\ell)} \otimes V_2^{(\ell)} \to V_2^{(\ell)} \otimes V_1^{(\ell)}$$

where $V^{(\ell)}$ is the $\frac{\ell}{2}$ spin representation.

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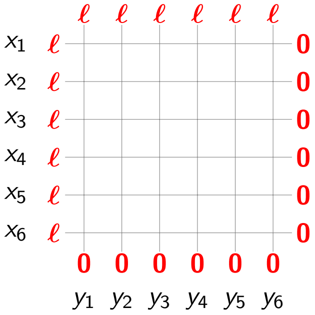

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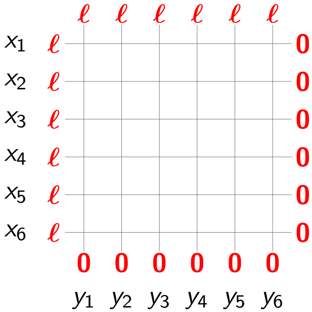

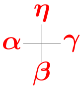

$$\alpha + \beta = \gamma + \eta$$

Fusion

We can get $V^{(\ell)}$ by projecting the product $V^{(1)} \otimes \ldots \otimes V^{(1)}$.

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Example ($\ell = 3$)

$$|3\rangle = |1\rangle \otimes |1\rangle \otimes |1\rangle$$
$$|2\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle \otimes |0\rangle$$

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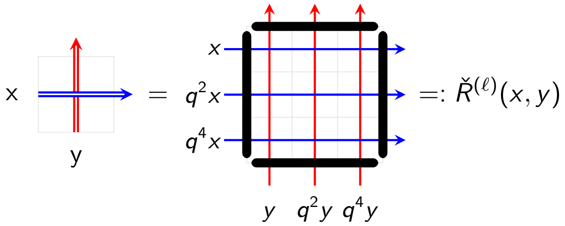
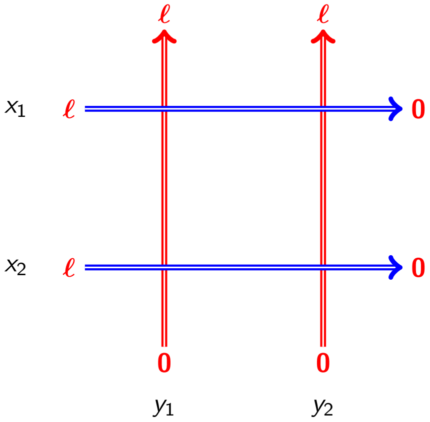
Define the new $\check{R}^{(\ell)}$ matrix by:



This satisfies Yang–Baxter equation and the Identity equation.

The Partition Function

We construct the partition function in the same way:



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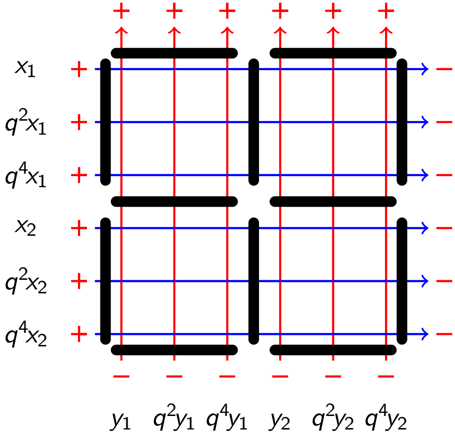

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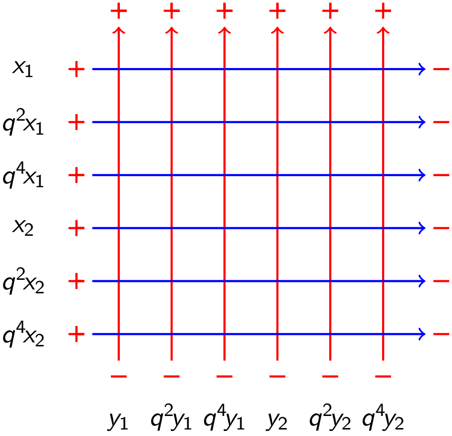

An Explicit Formula for the Partition Function

Let

$$\bar{x} = \{x_1, q^2 x_1, \ldots, q^{2\ell-2} x_1, \ldots, x_n, q^2 x_n, \ldots, q^{2\ell-2} x_n\}$$
$$\bar{y} = \{y_1, q^2 y_1, \ldots, q^{2\ell-2} y_1, \ldots, y_n, q^2 y_n, \ldots, q^{2\ell-2} y_n\}$$

Caradoc, Foda and Kitanine showed that:

$$\mathcal{Z}_{n,\ell}(x, y) = (\text{Pre-factor}) \det \left|\mathcal{A}(\bar{x}_i, \bar{y}_j)\right|_{i,j}^{\ell n \times \ell n}$$

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Properties

- *Satisfies a weaker recursion relation;*
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Wheel Condition

When $q^{2\ell+1} = 1$, the partition function satisfies the wheel condition:

Definition (Wheel condition)

A polynomial $P(z)$ staisfies the wheel condition if:

$$P(z) = 0 \qquad \text{if } z_k = q^{1+2s_2} z_j = q^{2+2s_1+2s_2} z_i$$

for all $s_1, s_2 \in \mathbb{N}_0$ such that $s_1 + s_2 \leq l - 1$ and for any choice $i < j < k$.

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Example ($\ell = 2$)

Impose $q^5 = 1$. It means that the polynomial vanishes whenever:

$$z_k = qz_j = q^2 z_i \qquad z_k = q^3 z_j = q^4 z_i \qquad z_k = qz_j = q^4 z_i$$

Wheel Condition

When $q^{2\ell+1} = 1$, the partition function satisfies the wheel condition:

Definition (Wheel condition)

A polynomial $P(z)$ staisfies the wheel condition if:

$$P(z) = 0 \qquad \text{if } z_k = q^{1+2s_2} z_j = q^{2+2s_1+2s_2} z_i$$

for all $s_1, s_2 \in \mathbb{N}_0$ such that $s_1 + s_2 \leq l - 1$ and for any choice $i < j < k$.

Example ($\ell = 2$)

Impose $q^5 = 1$. It means that the polynomial vanishes whenever:

$$z_k = qz_j = q^2 z_i \qquad z_k = q^3 z_j = q^4 z_i \qquad z_k = qz_j = q^4 z_i$$

We call this point $q^{2\ell+1} = 1$, the combinatorial point.

Macdonald Polynomials

Definition (Macdonald polynomials)

They are symmetric polynomials which depends in two parameters $(\tilde{q}, t)$.

$$0 = \langle P_\lambda(z;\tilde{q},t), P_\mu(z;\tilde{q},t)\rangle_{\tilde{q},t} \qquad \text{if } \lambda \neq \mu$$

$$P_\lambda(z;\tilde{q},t) = m_\lambda(z) + \textstyle\sum_{\mu\prec\lambda} c_{\lambda,\mu}(\tilde{q},t) m_\mu(z)$$

where the inner product is defined by:

$$\langle p_\lambda(z), p_\mu(z)\rangle_{\tilde{q},t} = z_\mu \delta_{\lambda\mu} \prod_i \frac{1-\tilde{q}^{\mu_i}}{1-t^{\mu_i}}$$

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Theorem (Jimbo, Miwa, Feigin and Mukhin)

The symmetric polynomials obeying to the wheel condition are spanned by the Macdonald polynomials P_λ (with $\tilde{q} = q^2$ and $t = q$), such that:

$$\lambda_i - \lambda_{i+2} \geq \ell$$

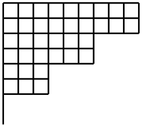
The Partition Function as a Macdonald Polynomial

Fix the number of boxes to $\ell n(n-1)$, then we have no choice.

$$Y_{n,\ell} = \{\ell(n-1), \ell(n-1), \ell(n-2), \ell(n-2), \ldots, \ell, \ell, 0, 0\}$$

The Partition Function as a Macdonald Polynomial

Fix the number of boxes to $\ell n(n-1)$, then we have no choice.

$Y_{4,3} =$ 

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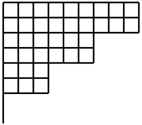
$Y_{4,3} =$ 

Theorem (Main result)

At the combinatorial point, the partition function is the Macdonald polynomial:

$$\mathcal{Z}_{n,\ell}(x,y) \propto P_{Y_{n,\ell}}(x,y;q^2,q)$$

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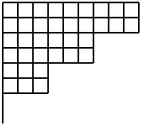
$Y_{4,3} =$ 

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Corollary (Symmetry)

The partition function is a fully symmetric homogeneous polynomial.

Open questions

- The 6–Vertex model is in bijection with Alternating Sign Matrices.
  - Can this generalization lead us to nice combinatorics?
  - What about the homogeneous limit?
- Wheel conditions and determinants.
  - Can we obtain different wheel conditions, with similar methods?
  - And the same wheel condition but higher degree?
- Can we prove directly that the partition function is fully symmetric?

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