I consider two sets of objects that apriori know nothing about each other.

Set 1

Consider the planar limit of 4D N=4 super Yang-Mills.

1 spin-1 gauge field,
4 spin-1/2 fermions, and
6 spin-0 real scalars

The 6 real scalars can be written as 3 charged scalars {X, Y, Z} and their complex conjugates {Xc, Yc, Zc}

They live in the adjoint rep of the gauge group su(n), so they can be represented as n-by-n matrices in colour space.

Focus on any su(2) scalar sub sector, such as {X, Y}

Consider the composite operators, made of an arbitrary number of X and Y scalar, such that

1. They are in the form of linear combinations of terms. Each term is a single trace of a product of X and Y fields, Tr[XXYXYYXXXYY....], such that the product has L terms, and the number of X fields only is N

2. They are eigenstates of the dilatation operator, or equivalently, the mixing matrix M

Set 2

Consider a length-L periodic homogeneous XXX spin-1/2 chain.
Consider the N-magnon states. They are labeled by N auxiliary-space rapidity variables.

If the rapidity variables are generic (unconstrained by any conditions), the state is an "off-shell" state.

If the rapidity variables obey the Bethe Ansatz equations, the state is an "on-shell" state (an eigenstate of the spin chain transfer matrix).

The elements of set 2 are the on-shell states, on a length-L lattice, with N magnons.

A bijection (Minahan and Zarembo, 2002)

There is a 1-to-1 correspondence between the elements of set 1 and set 2

The mixing matrix M (the non-classical part of the dilatation operator) is identical to the Hamiltonian H of the spin chain.

Every eigenstate O of the mixing matrix M,
that is a product of L fields, with N x-fields,
maps to
an eigenstate E of the spin chain Hamiltonian H,
of a length-L spin-chain, with N magnons.

The anomalous conformal dimension of O is equal to the eigenvalue e corresponding to the eigenstate E.

Remarks

Minahan and Zarembo were far from the first to work on the connection between gauge theories and integrable spin chains. Lipatov, Faddeev and Korchemsky, and others from the late '80's, and early 90's

The bijection of Minahan and Zarembo extends to the entire super YM theory, with suitable extensions of the corresponding spin chain.

The bijection of M and Z is only the first in a series that contains a bijection of Nekrasov and Shatashvili that maps spin-chain Bethe on-shell states to ground states of N=2 super YM theories, and back.

New bijections, that connect the above objects to other objects that are far from obviously related to them, appear nowadays on a regular basis.

See the recent works of S Franco, M Yamazakie, D Xie, as well as J Teschner, A Goncharev and R Kenyon, and many others, and references therein.

So what? What is the use of the Minahan-Zarembo bijection?

It allows us to map problems that we want to solve on the gauge theory side to problems on integrable spin chains where they can be solved using the methods of quantum integrable models, such as the algebraic Bethe Ansatz.

- Problems that we wish to solve include

Compute the two point functions of local composite operators: The anomalous conformal dimensions.

This problem has basically been solved.

Compute the three-point functions of local composite operators: The structure constants.

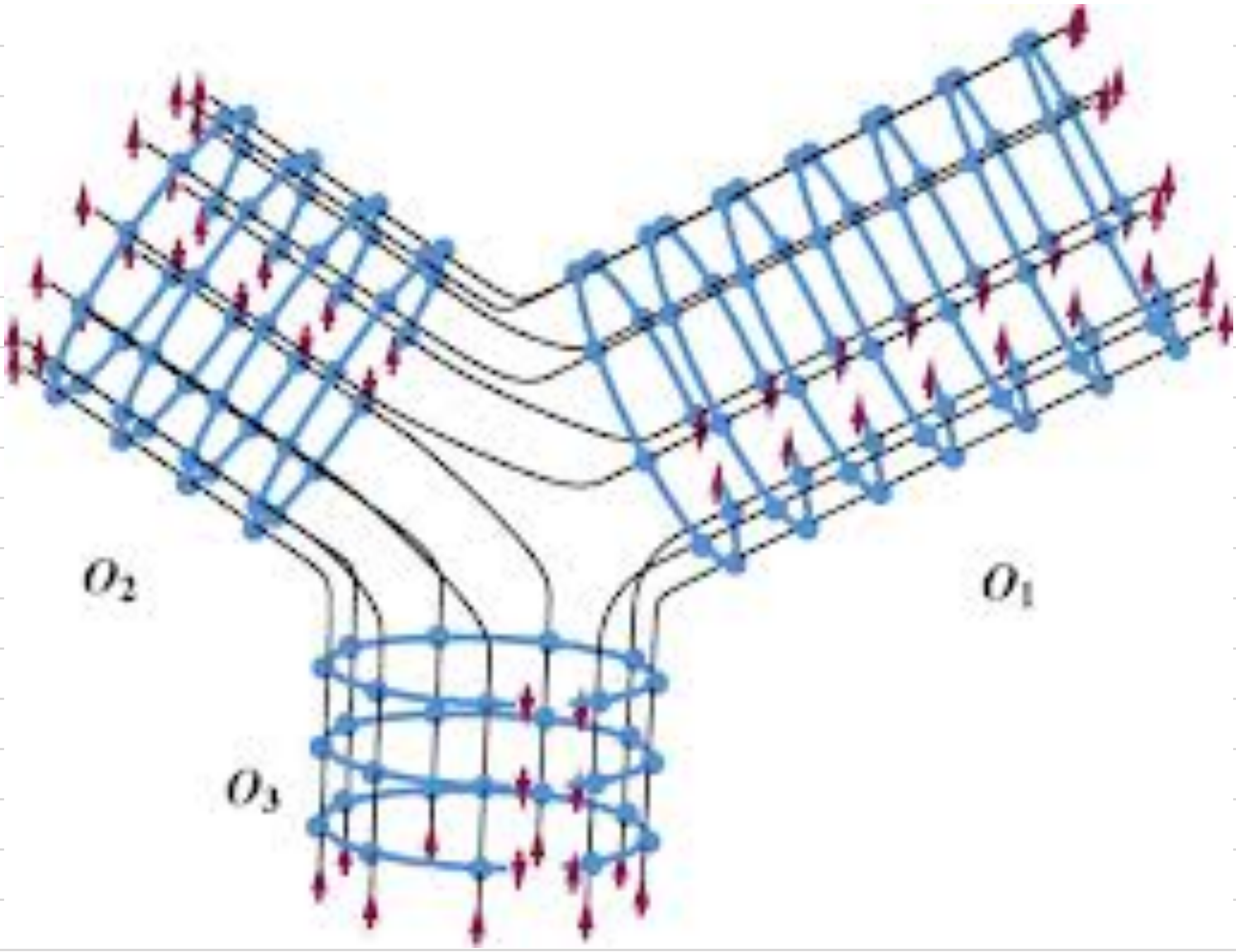
Working in the context of su(2) scalar subsectors, and at tree-level in the 't Hooft coupling constant, the structure constants can be mapped to a slight variation on Slavnov's scalar product, so they can be computed in determinant form, and the problem is solved.

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}}_j(y)\rangle = \delta_{ij}\mathcal{N}_i|x-y|^{-2\Delta_i}$$

where $\overline{\mathcal{O}}_i$ is the Wick conjugate of $\mathcal{O}_i$, Δ_i is the conformal dimension of $\mathcal{O}_i$ and $\mathcal{N}_i$ is a normalization factor. Later, we choose $\mathcal{N}_i$ to be (the square root of) the Gaudin norm of the corresponding spin-chain state.

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\rangle = \Big(\mathcal{N}_i\,\mathcal{N}_j\,\mathcal{N}_k\Big)^{1/2}\frac{C_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}|x_{jk}|^{\Delta_j+\Delta_k-\Delta_i}|x_{ki}|^{\Delta_k+\Delta_i-\Delta_j}}$$

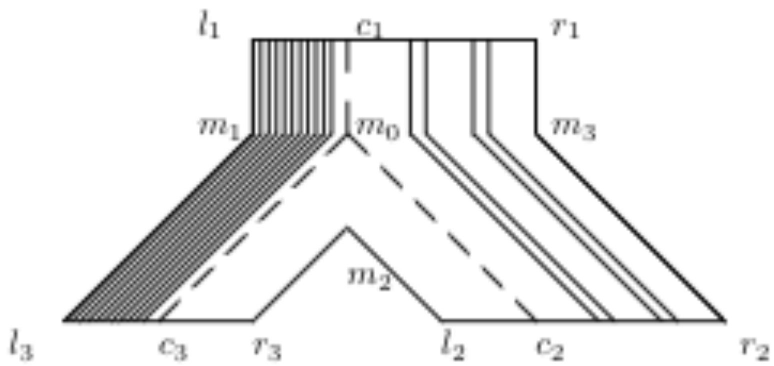
Here is a 3-dimensional sketch of the 3-point function of composite operators (Courtsy of N Gromov and P Vieira)



Think of each composite state as an L-site periodic lattice.
The black lines are worldlines of lattice sites.
The blue lines represent the action of the Bethe Ansatz creation operators on the initial reference state. They prepare the single-trace states that goes on to interact with one another.

Following Escobedo, Gromov, Sever and Vieira, the 3-point function involves TWO types of propagators

X-Xc propagators that we denote by black,lines, and y-yc propagators that we denote by white lines.



A schematic representation of a 3-point function. State $\mathcal{O}_1$ is on top. $\mathcal{O}_2$ and $\mathcal{O}_3$ are below, to the right and to the left.

Remarks

1. The computation of the structure constants of the su(2) local composite operators was set up by J Escobedo, N Gromov, A Sever and P Vieira, in a series of papers in 2010 and 2011.

2. Escobedo et al. set up the problem in a clear form and provided a partial solution in terms of a sum using the Izergin's-Korepin sum expression for the scalar product of two off-shell states. They did not use the Bethe equations satisfied by the rapidities.

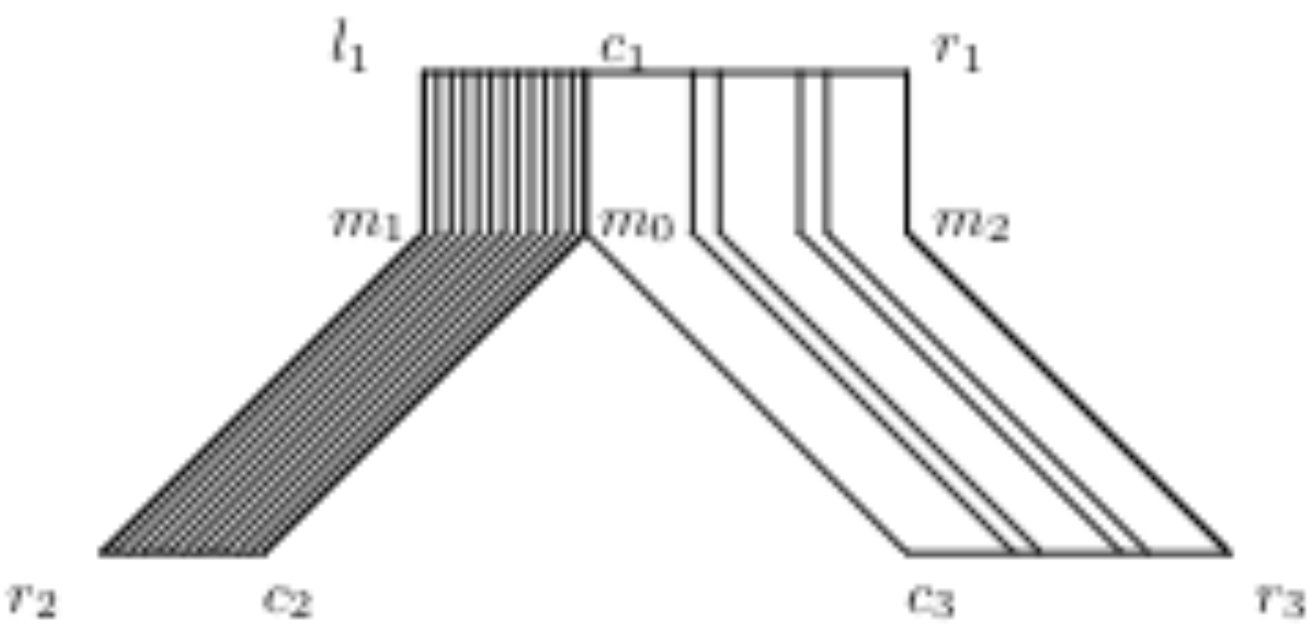
What I wish to do next is to formulate things in such a way that I can use the Bethe equations satisfied by the rapidities of state O1.

That will allow me to use Slavnov's determinant formula to evaluate the structure constants described earlier.

There are two important simplifications to make.

1. All propagators between O2 and O3 are white. They are y-yc propagators.

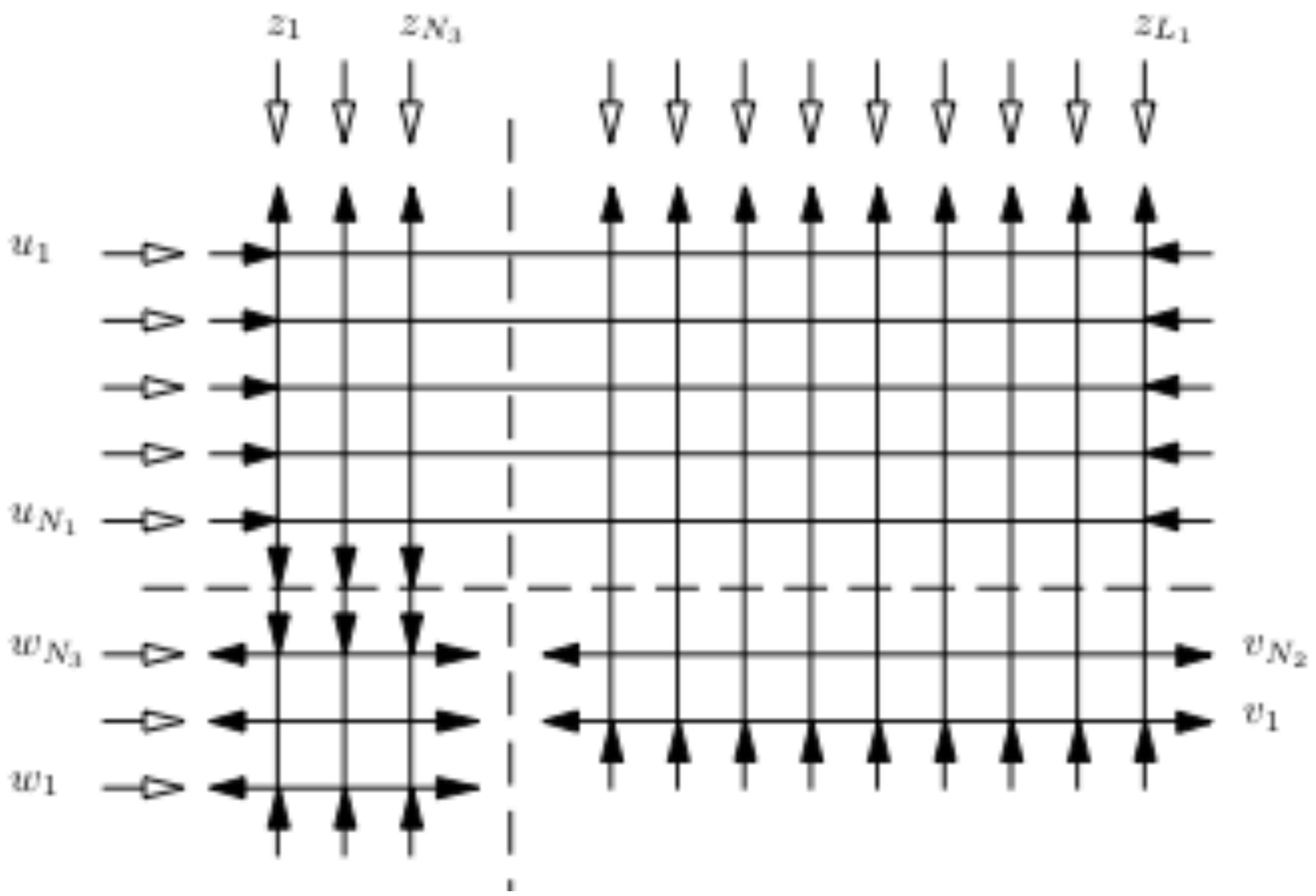
In spin chain language, they correspond to the inner product of a reference state and a reference state. The contribution of the part of the structure constant is 'frozen' and factorizes. It is normalised to 1



2. All propagators between O2 and O1 are black. They are x-Xc propagators.

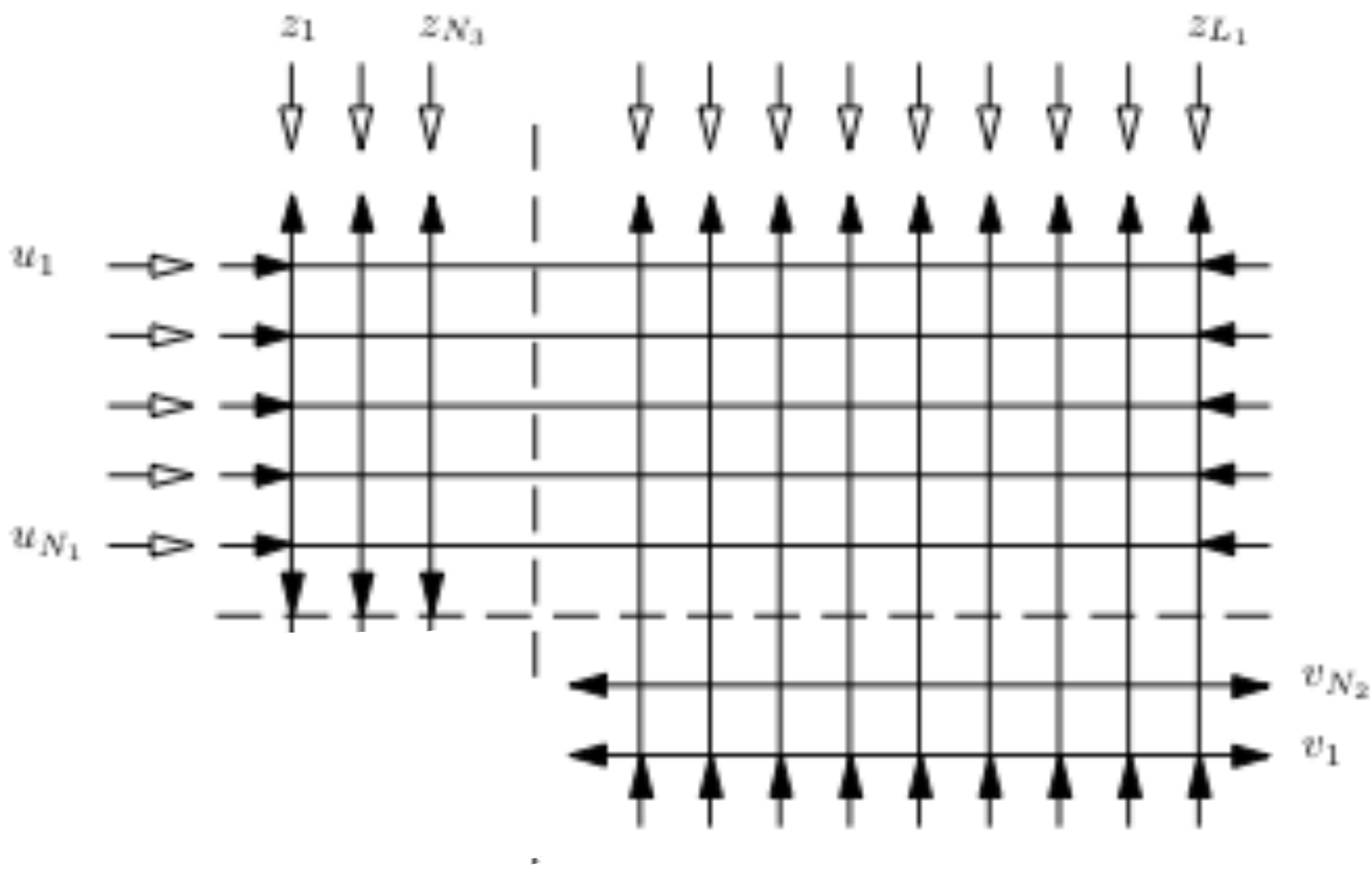
In spin chain language, they correspond to the inner product of a dual ref state and a dual ref state that is obtained from a ref state by the action of B operators with finite rapidities. The contribution of the part of the structure constant is 'frozen' and factorizes. One can show that it is a domain wall partition function.

In six-vertex model terms, we end up with the following configurations



Note the domain wall configuration in the lower left corner.

Removing the domain wall configurations, we obtain

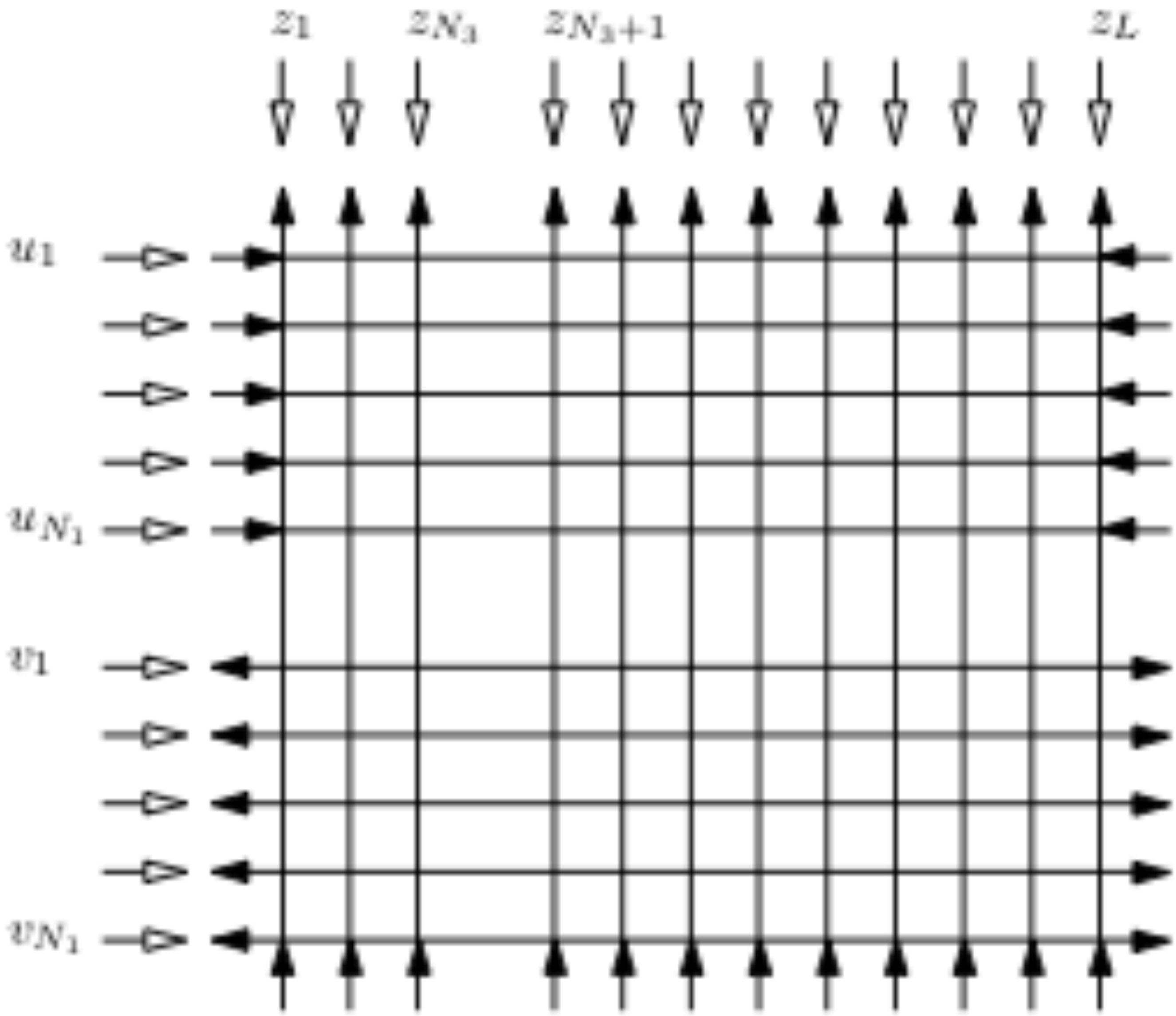


What is the partition function that corresponds to the above configuration?

It turns out to be a slight variation on Slavnov's scalar product of an on-shell state and an off-shell state.

Remark
In the following, I make use of a trick that I learnt from a paper by M Wheeler, that I believe he learnt from a paper by N Kitanine et co. Precise references are in my paper and recent joint papers with M Wheeler.

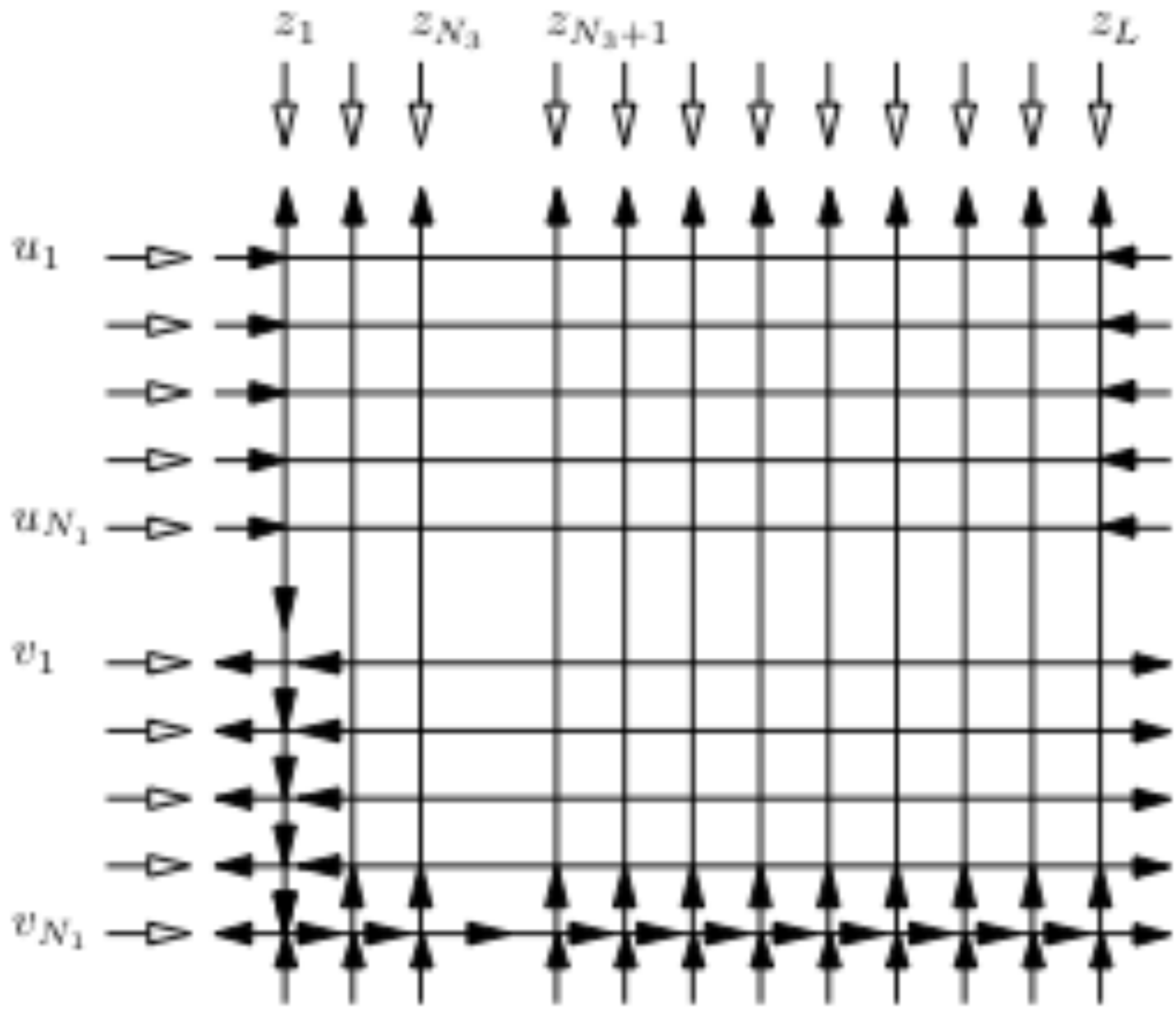
The configuration whose partition function is
Slavnov's scalar product is



The lower left corner can be either a type-b vertex with weight $(x - y)$, or a type-c vertex of constant weight, in some normalisation.

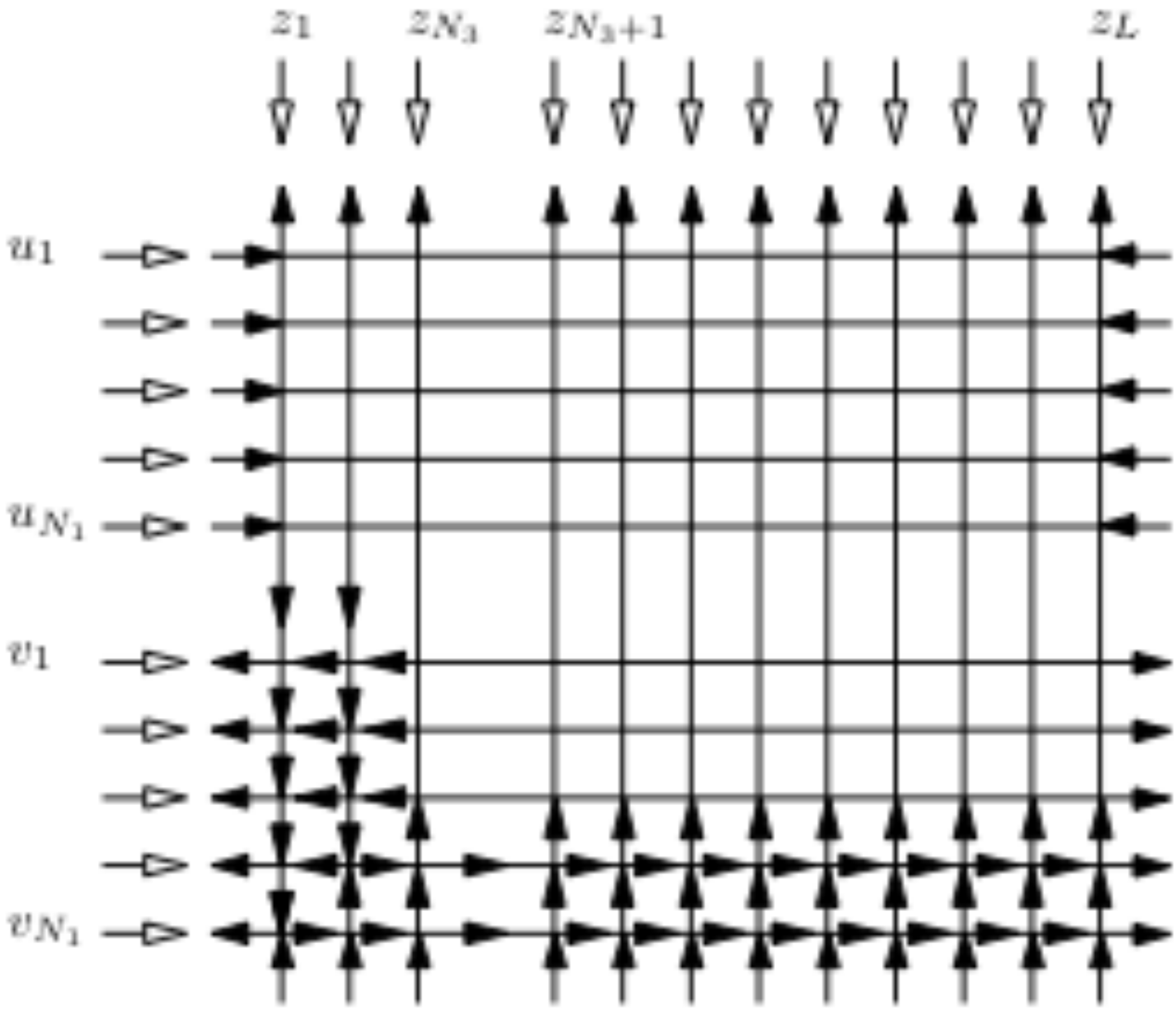
x is the auxiliary space rapidity, and
y is the in homogeneity variable, that flow through the vertex.

Choosing the auxiliary rapidity and the inhomogeneity variable that flow through the lower left vertex appropriately, we can set the weight of a type-b vertex at this position to zero, so only a type-c vertex is allowed.

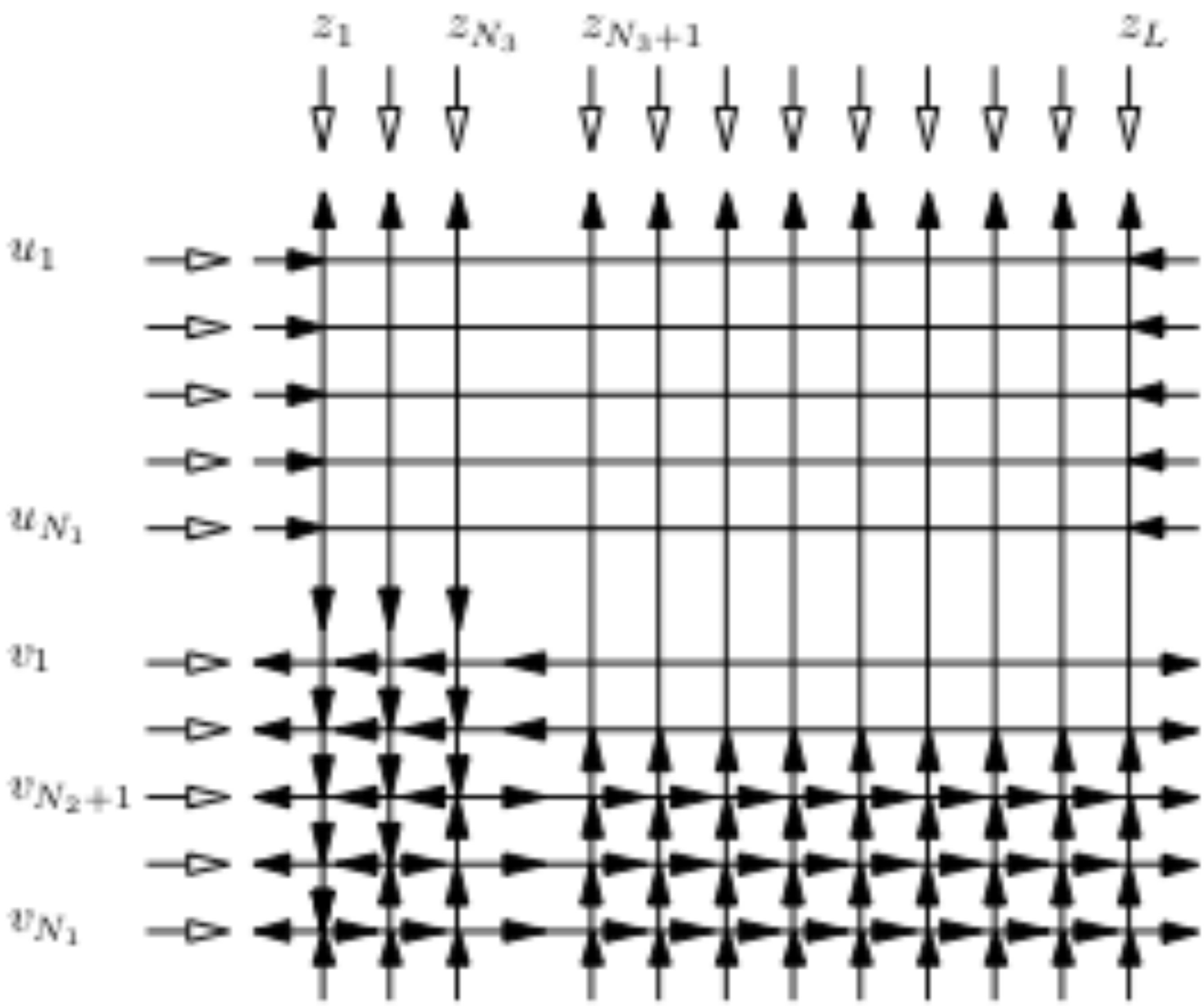


This "freezes" the lower row of vertices and part of the left column, as shown in the figure.

Repeating the exercise a second time, we obtain

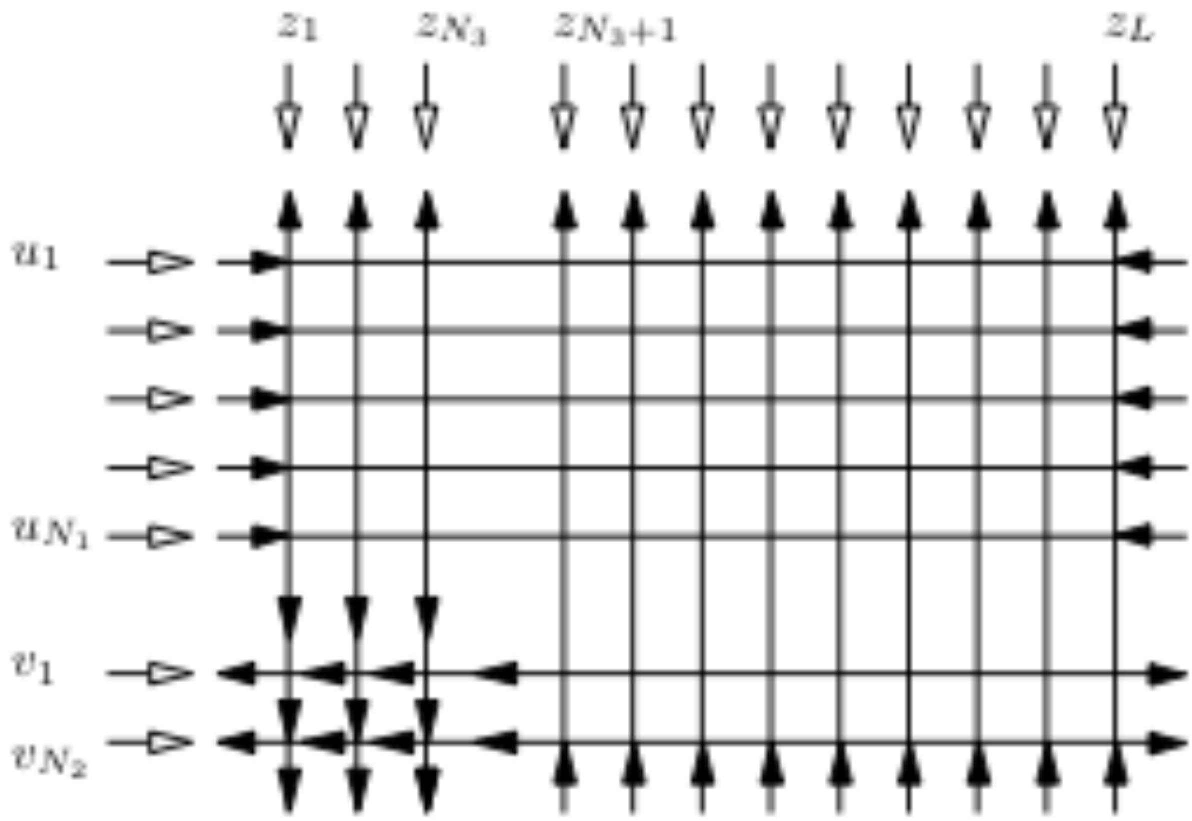


and one more time



we finally get the point, which is that up to frozen pieces that trivially factorizes, we are generating the object that we saw earlier, and that we need to compute the structure constants.

For example, removing the lower frozen rows, we obtain



Next, we remove the frozen 2-by-3 rectangle in the lower left corner, and account for its weight.

So, finally, now we know how to compute the structure constants. We start from Slavnov's scalar product, choose some of the free rapidities appropriately, and account for the weights of the frozen pieces, as well as other straightforward factors.

From the Kitanine et co, as well as M Wheeler, we know the expression for this "restricted" Slavnov scalar product.

A determinant expression for the Slavnov scalar product $S[L, N_1, N_2]$**.** Following [32, 33], we consider the $(N_1 \times N_1)$ matrix

$$\mathcal{S}\left(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L\right) = \begin{Bmatrix} f_1(z_1) & \cdots & f_1(z_{N_3}) & g_1(v_{N_2}) & \cdots & g_1(v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{N_1}(z_1) & \cdots & f_{N_1}(z_{N_3}) & g_{N_1}(v_{N_2}) & \cdots & g_{N_1}(v_1) \end{Bmatrix}$$

whose entries are the functions

$$f_i(z_j) = \left(\frac{[\eta]}{[u_{\beta_i} - z_j + \eta][u_{\beta_i} - z_j]}\right) \prod_{k=1}^{N_2} \frac{1}{[v_k - z_j]}$$

$$g_i(v_j) = \left\{\frac{[\eta]}{[u_{\beta_i} - v_j]}\right\} \times \left\{\left\{\prod_{k=1}^{L} \frac{[v_j - z_k + \eta]}{[v_j - z_k]} \prod_{k \neq i}^{N_1} [u_{\beta_k} - v_j + \eta]\right\} - e^{-2i\theta} \prod_{k \neq i}^{N_1} [u_{\beta_k} - v_j - \eta]\right\}$$

and where $N_3 = N_1 - N_2$. Since the auxiliary rapidities $\{u_\beta\}_{N_1}$ satisfy Bethe equations (35), following [32, 33] it is possible to show that

$$S[L, N_1, N_2] = \frac{\prod_{i=1}^{N_1} \prod_{j=1}^{N_3} [u_{\beta_i} - z_j + \eta] \det \mathcal{S}\left(\{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L\right)}{\prod_{1 \leq i < j \leq N_1} [u_{\beta_j} - u_{\beta_i}] \prod_{1 \leq i < j \leq N_2} [v_i - v_j] \prod_{1 \leq i < j \leq N_3} [z_i - z_j]}$$

where Ni is the number of magnons in state Oi,
$i = \{1, 2, 3\}$

Remarks

1. We can repeat the exercise in YM theories with less than maximal supersymmetries including pure QCD (without matter, gluondynamics). Joint works with M Wheeler and with Ch Ahn and R Nepomechie.

2. These determinants are discrete KP tau functions in their free parameters which play the role of Miwa variables. More precisely, they are Casoratians in these parameters.

The message is, when looking for determinant expressions, the right form to look for is the Casoratian form (think of a Wronskian).

3. Slavnov's scalar products are N-by-N determinants, where N id the number of magnons in either state. The "restricted" objects are N1-by-N1 determinants, where N1 is the number of magnons in O1, which is the number of magnons N in the original unrestricted object.

4. Following I Kostov and F Smirnov, we now know that Slavnov's determinant can be regarded as a special case of Izergin's determinant just as Izergin's is a special case of Slavnov's! These two objects have a "bootstrap-like" relationship to one another!

This leads to two new forms for the scalar product:
As a 2N-by-2N determinant (Kostov and Matsuo),
and as an L-by-L determinant (with Wheeler).

Every form is useful for something, and shows a
different aspect of this object.

Determinants in stat mech and the objects that they represent are mysterious and have many properties that we don't know.

The su(2) case at tree-level is so mysterious, but we know that they can be extended to higher rank because N=4 SYM had better be integrable!