Quantum steady states out of equilibrium

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Physical situation

$t \leq t_0$:

$\beta_l$

$R/2$

$t = t_0$:

$\beta_l$

$R$

$t > t_0$:

$\beta_r$

$R/2$

$R \gg v_F(t - t_0) \gg$ observation length, microscopic scales
Physical situation

\[ \langle \cdots \rangle_{\text{ness}} = \lim_{t_0 \to -\infty} \lim_{R \to \infty} \frac{\text{Tr} \left( e^{iHt_0} \rho_0 e^{-iHt_0} \cdots \right)}{\text{Tr} \left( \rho_0 \right)} \]

\[ \rho_0 = e^{-\beta_l H^l - \beta_r H^r} \]

\[ H = H^l + H^r + H_{\text{contact}} \]

Observables supported on a finite region
Description of the steady state

\[
\langle \ldots \rangle_{\text{ness}} = \frac{\text{Tr} \left( e^{-Y} \ldots \right)}{\text{Tr} \left( e^{-Y} \right)}
\]

Operator \( Y \):

- Commutes with the Hamiltonian \( H \)
- «Asymptotically looks like» \( \beta_l H^l + \beta_r H^r \)

★ Formally similar to a proposition, in the case of charge flow rather and energy flow, of Hershfield (PRL 1993)
★ Connected to scattering theory for nonequilibrium steady states
Description of the steady state

\[ Y = \beta_l \int_0^\infty d\theta E_\theta A(\theta)\dagger A(\theta) + \beta_r \int_{-\infty}^0 d\theta E_\theta A(\theta)\dagger A(\theta) \]

Total energy of right-moving asymptotic particles

Total energy of left-moving asymptotic particles

State

back in time...

[XY model: Aschaber & Pillet, 2003]
Energy current in CFT

\[ J = \frac{\pi c}{\sqrt{2}} (\beta_l^{-2} - \beta_r^{-2}) = \frac{\pi ck_B^2}{12\hbar} (T_l^2 - T_r^2) \]

central charge

\[ H^l = \int_{-R/2}^{0} dx \left( h_+^l(x) + h_-^l(x) \right) \]
\[ H^r = \int_{0}^{R/2} dx \left( h_+^r(x) + h_-^r(x) \right) \]

\[ T(x) = -\frac{c}{24} + \sum_{n \in \mathbb{Z}} L_n e^{-\frac{2\pi inx}{R}} \]

Virasoro
Energy current in CFT

Time evolution:

\[ h_\pm(x) = \begin{cases} 
  h^l_\pm(x) & (x < 0) \\
  h^r_\pm(x) & (x > 0) 
\end{cases} \Rightarrow e^{iHt} h_\pm(x) e^{-iHt} = h_\pm(x \mp t) \]

Gives rise to separation of right- and left-movers:

\[ J = \langle h_+ (x) - h_-(x) \rangle_{\text{ness}} \]

\[ = \lim_{R \to \infty} \frac{\text{Tr} \left( e^{-Y_R} (\tilde{h}_+ (x) - \tilde{h}_-(x)) \right)}{\rho_0 \text{Tr} (e^{-Y_R})} \]

Leading to a different representation of the algebra of observables:

\[ \tilde{h}_+ (x) = \frac{2\pi}{R^2} T(x), \quad \tilde{h}_- (x) = \frac{2\pi}{R^2} \bar{T}(x) \]

\[ Y_R = \frac{2\pi \beta_l}{R} L_0 - \frac{2\pi \beta_r}{R} \bar{L}_0 \]
Energy current in CFT

Hence:

\[ J = f(\beta_l) - f(\beta_r), \quad f(\beta) = -\lim_{R \to \infty} \frac{1}{R} \frac{d}{d\beta} \log Z(\beta) \]

where

\[ Z(\beta) = \text{Tr} \left( e^{-\frac{2\pi \beta}{R} L_0} \right) \]

and we can use

\[ Z(\beta) \sim N e^{\frac{\pi c R}{12\beta}} \]
Energy transfer fluctuations in CFT

We want to measure the fluctuations of the transfer of energy, whose «charge» can be taken as:

\[ Q = \frac{1}{2} (H^l - H^r) \]

\[ Q = q_0 \]

\[ Q = q_0 + q \]
Energy transfer fluctuations in CFT

\[ P(q, t) = \sum_{q_0} \text{Tr} \left( P_{q_0} + q e^{-iHt} P_{q_0} \rho_{\text{ness}} P_{q_0} e^{iHt} P_{q_0} + q \right) \]

\[ P(\lambda, t) = \sum_q e^{i\lambda q} P(q, t) \]

\[ \log P(\lambda, t) \sim tF(\lambda) + O(1) \]

Cumulant generating function

\[ = -i\lambda J + \ldots \]
charge transfer fluctuations in free-fermion systems

Lesovik-Levitov formula, 1993 & 1994 (also: Klich, Schonhammer, DB & BD, . . .)

energy transfer fluctuations in harmonic chains

Saito & Dhar, 2007
Energy transfer fluctuations in CFT

Use \( \sum_q f(q)P_q = f(Q) \) and \( P_q \propto \int d\mu e^{i\mu(Q-q)} \)

\[ F(\lambda) = \lim_{t \to \infty} t^{-1} \log \left[ \lim_{t_0 \to -\infty} \lim_{R \to \infty} \int d\mu \right] \]

\[ \text{Tr} \left( \rho_0(t_0) e^{-i\left(\frac{\lambda}{2}+\mu\right)Q} e^{i\lambda Q(t)} e^{-i\left(\frac{\lambda}{2}-\mu\right)Q} \right) \]

\[ \frac{\text{Tr} \rho_0(t_0)}{\text{Tr} \rho_0(t_0)} \]

\( \ll \text{simplify} \):
Fluctuation relation

\[ F(\lambda) = F(i(\beta_l - \beta_r) - \lambda) \]

Equivalent to:

\[ P(q, t \to \infty) = e^{(\beta_l - \beta_r)q} P(-q, t \to \infty) \]

Such a relation was argued for with first measurement at \( t = t_0 \)

Jarzynski, Wojcik (PRL 2004); Nice review: Esposito, Harbola, Mukamel (RMP 2009); More rigorous proof: Andrieux, Gaspard, Monnai, Tasaki (2008); Basic ideas: Gallavoti, ...

CFT derivation [DB & BD, in preparation]
Factorization

\[ e^{i\lambda Q(t)} e^{-i\lambda Q} = e^{i\lambda Q} + i\lambda \int_0^t dx (h_-(x) - h_+(-x)) e^{-i\lambda Q} \]

Observable supported on a finite region

In the steady state:

\[ Q \mapsto \frac{\pi}{R} (L_0 - \bar{L}_0), \quad h_\pm \mapsto \tilde{h}_\pm \]

With Y-operator, get factorization:

\[ F(\lambda) = f(\lambda, \beta_l) + f(-\lambda, \beta_r) \]
Energy transfer fluctuations in CFT

Using dimensional analysis, unique solution to:

- **Factorization**  \( F(\lambda) = f(\lambda, \beta_l) + f(-\lambda, \beta_r) \)
- **Leading behaviour**  \( F(\lambda) = O(\lambda) \)
- **Fluctuation relation**
A stochastic interpretation

Independent Poisson processes for jumps of every energy $E$, positive or negative, with intensity

$$dE \ e^{-\beta_i E} \quad (E > 0)$$

$$dE \ e^{\beta_r E} \quad (E < 0)$$
Nonequilibrium correlation functions in the massive Ising model

Gelfand-Naiman-Segal construction:

$$\langle \cdots \rangle_{\text{ness}} = \frac{\text{Tr}\left(e^{-Y} \cdots\right)}{\text{Tr}\left(e^{-Y}\right)} = Y\langle \text{vac}| \cdots| \text{vac}\rangle^Y$$

$$Y\langle a|b\rangle^Y = \frac{\text{Tr}\left(e^{-Y} A^\dagger B\right)}{\text{Tr}\left(e^{-Y}\right)}$$

some convenient normalization

In the Ising model:

$$|\theta \cdots\rangle^Y_{\epsilon\ldots} \equiv q_\epsilon(\theta) A^\epsilon(\theta) \cdots, \quad \epsilon \in \{+,-,\} = \{|\uparrow,\}\}$$
Nonequilibrium correlation functions in the massive Ising model

Form factor expansion

\[
Y \langle \text{vac} | \sigma(x) \sigma(0) | \text{vac} \rangle^Y = \sum_{\text{states}} \left| Y \langle \text{vac} | \sigma(0) | \text{state} \rangle^Y \right|^2 e^{-ip_{\text{state}}x}
\]

\[
= \sum_{k} \sum_{\epsilon \ldots} \int \frac{d\theta \cdots}{k!} \left| Y \langle \text{vac} | \sigma(0) | \theta \cdots \rangle^Y \epsilon \cdots \right|^2 e^{-i\epsilon p_{\theta} + \cdots}x
\]

The spin field is a normal-ordered exponential of a bilinear in the free modes:

\[
\sigma = : \exp \left[ \int d\theta d\theta' \left( \frac{1}{2} A(\theta) A(\theta') \langle \text{vac} | \sigma | \theta', \theta \rangle + A^\dagger(\theta) A(\theta') \langle \theta | \sigma | \theta' \rangle + \frac{1}{2} A^\dagger(\theta) A^\dagger(\theta') \langle \theta, \theta' | \sigma | \text{vac} \rangle \right) \right] :
\]
Nonequilibrium correlation functions
in the massive Ising model

Exact form factors
[YC & BD, in preparation]

[Bugrij, 2000&2001]
[Fonseca & Zamolodchikov, 2003]
[BD, 2005, 2007]

Lattice on the cylinder
QFT on the circle
QFT at finite temperature

Given
\[ Y|\theta \cdots \rangle_{\epsilon \cdots} = (Y(\theta) + \cdots) |\theta \cdots \rangle_{\epsilon \cdots} \]

we obtain:
\[ Y \langle \text{vac} | \sigma(0) | \theta \cdots \rangle_{\epsilon \cdots} Y = (h_{\epsilon}(\theta) \cdots) \times \text{usual form factors} \]

\[ \sqrt{\frac{i}{2\pi}} e^{\frac{i\pi}{4}} \exp \left[ \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\sinh(\theta - \theta')} \log \left( \frac{1 + e^{-Y(\theta')}}{1 - e^{-Y(\theta')}} \right) \right] \]
Nonequilibrium correlation functions in the massive Ising model

Analytic structure of $h_+(\theta)$

\[
\sinh \left( \text{Re}(\theta) \right) = \frac{2\pi n}{\beta_r}
\]

\[
\sinh \left( \text{Re}(\theta) \right) = \frac{2\pi n}{\beta_t}
\]

Branch cut

Thermal poles with different spacings
Conclusion and perspectives

We have a general description of energy-flow nonequilibrium steady states in QFT and CFT from which some exact results can be obtained.

«Leg factors» give initial data for sinh-Gordon classical inverse scattering problem of correlation function [BD and A. Gamsa, 2008]

A natural conjecture in integrable spin chains:

\[ Y = -\beta_l \sum_{p\lambda > 0} E\lambda - \beta_r \sum_{p\lambda < 0} E\lambda \]