Tetrahedral Y–systems

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RAQIS, Angers , September 2012

Y–systems

Y–systems originated from TBA (Al. Zamolodchikov, 1991)

Y–system for a pair of Dynkin diagrams with incidence matrices A, A':

$$Y_{i,j-1,k}\ Y_{i,j+1,k} = \frac{\prod_n (1+Y_{n,j,k})^{A_{in}}}{\prod_m (1+Y_{i,j,m}^{-1})^{A'_{km}}}$$

$i = 1, \ldots, r, \quad k = 1, \ldots, r'$ — "2D–coordinates", $\quad j \in \mathbb{N}$ — "time"

Y–system of $(A_r, A_{r'})$ type:

$$Y_{i,j-1,k}\ Y_{i,j+1,k} = \frac{(1+Y_{i-1,j,k})(1+Y_{i+1,j,k})}{(1+Y_{i,j,k-1}^{-1})(1+Y_{i,j,k+1}^{-1})}$$

no denominator if $r' = 1$

periodicity: $Y_{i,j+p,k} = Y_{i,j,k}$

period: $p = 2(h + h') = 2(r + r' + 2)$

observed by Zamolodchikov (1991).

proofs for some cases:
Frenkel–Szenes (1995), Gliozzi–Tateo (1996), Fomin–Zelevinsky (2003), Volkov (2007), Szenes (2009), Kuniba–Nakanishi (2010)...

proof in the general case: Keller (2010).

Example: (A_2, A_1), $Y_{i,j-1}\ Y_{i,j+1} = 1 + Y_{i\pm1,j}$

$Y_{11} = x,\ \ Y_{22} = y,\ \ Y_{13} = \frac{1+y}{x},\ \ Y_{24} = \frac{1+x+y}{xy},\ \ Y_{15} = \frac{1+x}{y},$

$Y_{26} = x,\ \ \ Y_{17} = y,\ \ldots$

Dilogarithm identities

Rogers dilogarithm: $L(x) = \sum_{n\geq 1} \frac{x^n}{n^2} + \frac{1}{2} \log x \log(1-x)$

5–term relation: $L(x) + L(y) = L\Big(\frac{x(1-y)}{1-xy}\Big) + L(xy) + L\Big(\frac{y(1-x)}{1-xy}\Big)$

$l(x) \equiv L\big(\frac{x}{1+x}\big)$

(A_2, A_1): $l(Y_1) + l(Y_2) = l(Y_3^{-1}) + l(Y_4^{-1}) + l(Y_5^{-1})$

$(A_r, A_{r'})$: $\underbrace{\sum l(Y)}_{\frac{1}{2}\, r\, r'\, h' \text{ terms}} = \underbrace{\sum l(Y^{-1})}_{\frac{1}{2}\, r\, r'\, h \text{ terms}}$

(A_2, A_1): $2 \sim 3$

(A_2, A_2): $6 \sim 6$

(A_3, A_1): $6 \sim 3$

Quantum Y–systems and quantum dilogarithm

(A_2, A_1): if $Y_{11} \equiv x$ and $Y_{22} \equiv y$ and $yx = q^2 xy$, then

$$Y_{i,j-1}\ Y_{i,j+1} = 1 + q^{-1} Y_{i\pm1,j}$$

is periodic with $p/2 = 5$.

$Y_{11} = x,\ \ Y_{22} = y,\ \ Y_{13} = x^{-1}(1 + q^{-1}y),$
$Y_{24} = x^{-1}(q + x + y)y^{-1},\ \ Y_{15} = (1 + q^{-1}x)y^{-1},$
$Y_{26} = x$, $Y_{17} = y$, $\dots$

Volkov (2011):

$$\left[Y_1\right]_q\ \left[Y_2\right]_q = \left[Y_5^{-1}\right]_q \left[Y_4^{-1}\right]_q \left[Y_3^{-1}\right]_q$$

Quantum dilogarithm: $(q)_n \equiv (1-q)\dots(1-q^n)$,

$$[x]_q = \sum_{n\geq 0} (-1)^n \frac{x^n}{(q^2)_n}$$

Properties:

$$\left[q^2 x\right]_q = (1+x)\,[x]_q\,, \qquad [0]_q = 1$$

If $yx = q^2 xy$, then

$$[x]_q\ [y]_q = [x+y]_q$$

“pentagon” relation: $$[x]_q\ [x\,y]_q\ [y]_q = [y]_q\ [x]_q$$

"Tropical" Y–systems

The "limit" $Y \gg 1$ yields "tropical" Y–system (periodic with $p/2 = r+1$)

$$(A_r, A_{r'}) : \qquad \tilde{Y}_{i,j-1}\ \tilde{Y}_{i,j+1} = \tilde{Y}_{i-1,j}\tilde{Y}_{i+1,j}$$

$$(A_2, A_1) : \qquad \tilde{Y}_{i,j-1}\ \tilde{Y}_{i,j+1} = \tilde{Y}_{i\pm1,j}$$

$\tilde{Y}_1 = x$, $\tilde{Y}_2 = y^{-1}$, $\tilde{Y}_3 = x^{-1}y^{-1}$, $\tilde{Y}_4 = x^{-1}$, $\tilde{Y}_5 = y$ then

$$\left[\tilde{Y}_5\right]_q \left[\tilde{Y}_1\right]_q = \left[\tilde{Y}_4^{-1}\right]_q \left[\tilde{Y}_3^{-1}\right]_q \left[\tilde{Y}_2^{-1}\right]_q$$

holds by the pentagon relation.

The quantum dilogarithm identities for $\tilde{Y}$'s can be transformed into the relations for Y's.

How to construct tropical quantum dilogarithm identities for higher rank Y–systems?

Kashaev, Volkov (1998): two pentagons imply $4 \sim 4$ identity:

If $yx = q^2 xy$, $xz = q^2 zx$, $zy = q^2 yz$, then

$$T \equiv [x]_q \, [xy]_q \, [z]_q \, [y]_q = [y]_q \, [yz]_q \, [x]_q \, [z]_q$$

$\rho : x \to y, \; y \to z, \; z \to x$ is an automorphism of order three. Therefore

$$T = \rho(T) = \rho(\rho(T))$$

Recall: $[0]_q$=1. Hence

$$T \,\Big|_{z=0} = [x]_q \, [xy]_q \, [y]_q = [y]_q \, [x]_q$$

The quiver Q_r, $r \geq 2$ with $\binom{r}{2}$ vertices.

$$Q_3: \begin{tikzcd} (1,2) \arrow[r] & (1,3) \arrow[d] \\ & (2,3) \arrow[ul] \end{tikzcd} \qquad Q_4: \begin{tikzcd} (1,2) \arrow[r] & (1,3) \arrow[r] \arrow[d] & (1,4) \arrow[d] \\ & (2,3) \arrow[ul] \arrow[r] & (2,4) \arrow[ul] \arrow[d] \\ & & (3,4) \arrow[ul] \end{tikzcd}$$

$\mathcal{A}_r$ is a unital associative algebra with $\binom{r}{2}$ generators Z_{ij} assigned to the vertices of Q_r and the following defining relations:

$$\begin{aligned} \mathsf{Z}_{ij}\mathsf{Z}_{i'j'} &= q^2\, \mathsf{Z}_{i'j'}\mathsf{Z}_{ij} \qquad && \text{if } \ (i,j) \to (i',j'), \\ \mathsf{Z}_{ij}\mathsf{Z}_{i'j'} &= \mathsf{Z}_{i'j'}\mathsf{Z}_{ij} && \text{if } \ (i,j) \not\leftrightarrow (i',j') \end{aligned}$$

$\mathcal{A}_r$ possesses a third order automorphism ρ corresponding to the rotation of Q_r by $2\pi/3$.

$\mathcal{A}_r$ admits reductions: $\mathsf{Z}_{ij} \to 0$. In particular, $\mathcal{A}_r \subset \mathcal{A}_{r+1}$

The group $B(r)$ with generators R_{abc}, $1 \le a < b < c \le r$ and relations:

$$(*) \qquad R_{abc}R_{abd}R_{acd}R_{bcd} = R_{bcd}R_{acd}R_{abd}R_{abc}$$

and R_{abc} commutes with $R_{a'b'c'}$ unless they have exactly two common indices.

$(*)$ – tetrahedron equation – A. Zamolodchikov (1981).

$\vec{W}(r)$ – lexicographically ordered product of all generators. E.g.
$\vec{W}(4) = R_{123}R_{124}R_{134}R_{234}$
$\vec{W}(5) = R_{123}R_{124}R_{125}R_{134}R_{135}R_{145}R_{234}R_{235}R_{245}R_{345}$

$\overset{\leftarrow}{W}(r)$ – reversed order.

Theorem. $\qquad \vec{W}(r) = \overset{\leftarrow}{W}(r)$.

Define: $T(r) = \vec{W}(r)$ where $R_{abc} \to \left[\prod_{k=0}^{c-b-1} \mathsf{Z}_{a+k,b+k} \right]_q$

Identifying the argument $\prod_{k=0}^{c-b-1} \mathsf{Z}_{a+k,b+k}$ as $\tilde{Y}_{c-b,a+c,b-a}$,
we get a tropical A'_r Y–system:

$$\tilde{Y}_{i,j-1,k}\,\tilde{Y}_{i,j+1,k} = \tilde{Y}_{i-1,j,k}\,\tilde{Y}_{i+1,j,k}$$

Theorem. $T(r) = \rho(T(r))$

Proof: substitute in $\vec{W}(r) = \overset{\leftarrow}{W}(r)$ the Sergeev–Kashaev–Volkov solution of the TE:

$$R_{abc} \to \mathsf{F}_{abc}$$
$$R_{abc} \to \mathsf{F}_{abc} \cdot \left[\mathsf{x}_{ab}\mathsf{x}_{bc}^{-1}\mathsf{y}_{bc}^{-2}\right]_q$$

on the space of formal series in $r(r-1)/2$ variables x_{ab}, $1 \leq a < b \leq r$.

$$\mathsf{x}_{ab} : x_{ab}^m \to x_{ab}^{m+1}, \qquad \mathsf{y}_{ab} : x_{ab}^m \to q^m x_{ab}^m$$
$$\mathsf{F}_{abc} : x_{ab}^m x_{ac}^n x_{bc}^k \to x_{ab}^m x_{ac}^{k+m} x_{bc}^{n-m}$$

$\mathsf{F}^2 = id$. F is an analogue of P in R = PŘ

$T(r) = \rho(T(r))$ for $r = 4$:

$$[\mathsf{Z}_{12}]_q [\mathsf{Z}_{12}\mathsf{Z}_{23}]_q [\mathsf{Z}_{13}]_q [\mathsf{Z}_{23}]_q [\mathsf{Z}_{12}\mathsf{Z}_{23}\mathsf{Z}_{34}]_q [\mathsf{Z}_{13}\mathsf{Z}_{24}]_q [\mathsf{Z}_{23}\mathsf{Z}_{34}]_q [\mathsf{Z}_{14}]_q [\mathsf{Z}_{24}]_q [\mathsf{Z}_{34}]_q$$
$$=$$
$$[\mathsf{Z}_{14}]_q [\mathsf{Z}_{24}]_q [\mathsf{Z}_{34}]_q [\mathsf{Z}_{14}\mathsf{Z}_{13}]_q [\mathsf{Z}_{24}\mathsf{Z}_{23}]_q [\mathsf{Z}_{14}\mathsf{Z}_{13}\mathsf{Z}_{12}]_q [\mathsf{Z}_{13}]_q [\mathsf{Z}_{23}]_q [\mathsf{Z}_{13}\mathsf{Z}_{12}]_q [\mathsf{Z}_{12}]_q$$

Reduction $Z_{14} = 0$, $Z_{34} = 0$ yields (A_2, A_2) identity $(6 \sim 6)$:

$$\begin{aligned} &[\mathsf{Z}_{12}]_q [\mathsf{Z}_{12}\mathsf{Z}_{23}]_q [\mathsf{Z}_{13}]_q [\mathsf{Z}_{23}]_q [\mathsf{Z}_{13}\mathsf{Z}_{24}]_q [\mathsf{Z}_{24}]_q \\ &= [\mathsf{Z}_{24}]_q [\mathsf{Z}_{24}\mathsf{Z}_{23}]_q [\mathsf{Z}_{13}]_q [\mathsf{Z}_{23}]_q [\mathsf{Z}_{13}\mathsf{Z}_{12}]_q [\mathsf{Z}_{12}]_q \end{aligned}$$

Reduction $Z_{23} = 0$, $Z_{24} = 0$, $Z_{34} = 0$ yields (A_3, A_1) identity $(3 \sim 6)$:

$$[\mathsf{Z}_{12}]_q [\mathsf{Z}_{13}]_q [\mathsf{Z}_{14}]_q = [\mathsf{Z}_{14}]_q [\mathsf{Z}_{14}\mathsf{Z}_{13}]_q [\mathsf{Z}_{14}\mathsf{Z}_{13}\mathsf{Z}_{12}]_q [\mathsf{Z}_{13}]_q [\mathsf{Z}_{13}\mathsf{Z}_{12}]_q [\mathsf{Z}_{12}]_q$$

$$\boxed{(A_r, A_r) \longrightarrow (A'_r) \longrightarrow (A_{r_1}, A_{r_2}), \quad r_1 + r_2 = r}$$