

Tetrahedral Y-systems

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Y-systems

Y-systems originated from TBA (Al. Zamolodchikov, 1991)

Y-system for a pair of Dynkin diagrams with incidence matrices A, A' :

$$Y_{i,j-1,k} Y_{i,j+1,k} = \frac{\prod_n (1 + Y_{n,j,k})^{A_{in}}}{\prod_m (1 + Y_{i,j,m}^{-1})^{A'_{km}}}$$

$i = 1, \dots, r, \quad k = 1, \dots, r' — “2D-coordinates”, \quad j \in \mathbb{N} — “time”$

Y-system of $(A_r, A_{r'})$ type:

$$Y_{i,j-1,k} Y_{i,j+1,k} = \frac{(1 + Y_{i-1,j,k})(1 + Y_{i+1,j,k})}{(1 + Y_{i,j,k-1}^{-1})(1 + Y_{i,j,k+1}^{-1})}$$

no denominator if $r' = 1$

periodicity: $Y_{i,j+p,k} = Y_{i,j,k}$

period: $p = 2(h + h') = 2(r + r' + 2)$

observed by Zamolodchikov (1991).

proofs for some cases:

Frenkel–Szenes (1995), Gliozzi–Tateo (1996), Fomin–Zelevinsky (2003), Volkov (2007), Szenes (2009), Kuniba–Nakanishi (2010)...

proof in the general case: Keller (2010).

Example: (A_2, A_1) , $Y_{i,j-1} Y_{i,j+1} = 1 + Y_{i\pm 1, j}$

$Y_{11} = x$, $Y_{22} = y$, $Y_{13} = \frac{1+y}{x}$, $Y_{24} = \frac{1+x+y}{xy}$, $Y_{15} = \frac{1+x}{y}$,

$Y_{26} = x$, $Y_{17} = y$, ...

Dilogarithm identities

Rogers dilogarithm:
$$L(x) = \sum_{n \geq 1} \frac{x^n}{n^2} + \frac{1}{2} \log x \log(1-x)$$

5-term relation:
$$L(x) + L(y) = L\left(\frac{x(1-y)}{1-xy}\right) + L(xy) + L\left(\frac{y(1-x)}{1-xy}\right)$$

$$l(x) \equiv L\left(\frac{x}{1+x}\right)$$

$$(A_2, A_1): \quad l(Y_1) + l(Y_2) = l(Y_3^{-1}) + l(Y_4^{-1}) + l(Y_5^{-1})$$

$$(A_r, A_{r'}) : \quad \underbrace{\sum l(Y)}_{\frac{1}{2} r r' h' \text{ terms}} = \underbrace{\sum l(Y^{-1})}_{\frac{1}{2} r r' h \text{ terms}}$$

$$(A_2, A_1): \quad 2 \sim 3$$

$$(A_2, A_2): \quad 6 \sim 6$$

$$(A_3, A_1): \quad 6 \sim 3$$

Quantum Y -systems and quantum dilogarithm

(A_2, A_1) : if $Y_{11} \equiv x$ and $Y_{22} \equiv y$ and $yx = q^2xy$, then

$$Y_{i,j-1} Y_{i,j+1} = 1 + q^{-1}Y_{i\pm 1,j}$$

is periodic with $p/2 = 5$.

$$Y_{11} = x, \quad Y_{22} = y, \quad Y_{13} = x^{-1}(1 + q^{-1}y),$$

$$Y_{24} = x^{-1}(q + x + y)y^{-1}, \quad Y_{15} = (1 + q^{-1}x)y^{-1},$$

$$Y_{26} = x, \quad Y_{17} = y, \quad \dots$$

Volkov (2011):

$$[Y_1]_q [Y_2]_q = [Y_5^{-1}]_q [Y_4^{-1}]_q [Y_3^{-1}]_q$$

Quantum dilogarithm: $(q)_n \equiv (1 - q) \dots (1 - q^n)$,

$$[x]_q = \sum_{n \geq 0} (-1)^n \frac{x^n}{(q^2)_n}$$

Properties:

$$[q^2 x]_q = (1 + x) [x]_q, \quad [0]_q = 1$$

If $yx = q^2 xy$, then

$$[x]_q [y]_q = [x + y]_q$$

“pentagon” relation: $[x]_q [x y]_q [y]_q = [y]_q [x]_q$

“Tropical” Y -systems

The “limit” $Y \gg 1$ yields “tropical” Y -system (periodic with $p/2 = r + 1$)

$$(A_r, A_{r'}) : \quad \tilde{Y}_{i,j-1} \tilde{Y}_{i,j+1} = \tilde{Y}_{i-1,j} \tilde{Y}_{i+1,j}$$

$$(A_2, A_1) : \quad \tilde{Y}_{i,j-1} \tilde{Y}_{i,j+1} = \tilde{Y}_{i\pm 1,j}$$

$\tilde{Y}_1 = x$, $\tilde{Y}_2 = y^{-1}$, $\tilde{Y}_3 = x^{-1}y^{-1}$, $\tilde{Y}_4 = x^{-1}$, $\tilde{Y}_5 = y$ then

$$\left[\tilde{Y}_5 \right]_q \left[\tilde{Y}_1 \right]_q = \left[\tilde{Y}_4^{-1} \right]_q \left[\tilde{Y}_3^{-1} \right]_q \left[\tilde{Y}_2^{-1} \right]_q$$

holds by the pentagon relation.

The quantum dilogarithm identities for \tilde{Y} 's can be transformed into the relations for Y 's.

How to construct tropical quantum dilogarithm identities for higher rank Y -systems?

Kashaev, Volkov (1998): two pentagons imply $4 \sim 4$ identity:

If $yx = q^2xy$, $xz = q^2zx$, $zy = q^2yz$, then

$$T \equiv [x]_q [xy]_q [z]_q [y]_q = [y]_q [yz]_q [x]_q [z]_q$$

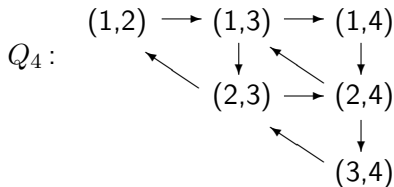
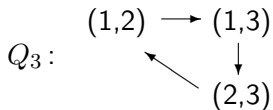
$\rho : x \rightarrow y, y \rightarrow z, z \rightarrow x$ is an automorphism of order three. Therefore

$$T = \rho(T) = \rho(\rho(T))$$

Recall: $[0]_q = 1$. Hence

$$T \Big|_{z=0} = [x]_q [xy]_q [y]_q = [y]_q [x]_q$$

The quiver Q_r , $r \geq 2$ with $\binom{r}{2}$ vertices.



\mathcal{A}_r is a unital associative algebra with $\binom{r}{2}$ generators Z_{ij} assigned to the vertices of Q_r and the following defining relations:

$$\begin{aligned}
 Z_{ij}Z_{i'j'} &= q^2 Z_{i'j'}Z_{ij} && \text{if } (i,j) \rightarrow (i',j'), \\
 Z_{ij}Z_{i'j'} &= Z_{i'j'}Z_{ij} && \text{if } (i,j) \not\rightarrow (i',j')
 \end{aligned}$$

\mathcal{A}_r possesses a third order automorphism ρ corresponding to the rotation of Q_r by $2\pi/3$.

\mathcal{A}_r admits reductions: $Z_{ij} \rightarrow 0$. In particular, $\mathcal{A}_r \subset \mathcal{A}_{r+1}$

The group $B(r)$ with generators R_{abc} , $1 \leq a < b < c \leq r$ and relations:

$$(*) \quad R_{abc}R_{abd}R_{acd}R_{bcd} = R_{bcd}R_{acd}R_{abd}R_{abc}$$

and R_{abc} commutes with $R_{a'b'c'}$ unless they have exactly two common indices.

(*) – tetrahedron equation – A. Zamolodchikov (1981).

$\vec{W}(r)$ – lexicographically ordered product of all generators. E.g.

$$\vec{W}(4) = R_{123}R_{124}R_{134}R_{234}$$

$$\vec{W}(5) = R_{123}R_{124}R_{125}R_{134}R_{135}R_{145}R_{234}R_{235}R_{245}R_{345}$$

$\overleftarrow{W}(r)$ – reversed order.

Theorem. $\vec{W}(r) = \overleftarrow{W}(r)$.

Define: $T(r) = \vec{W}(r)$ where $R_{abc} \rightarrow \left[\prod_{k=0}^{c-b-1} Z_{a+k, b+k} \right]_q$

Identifying the argument $\prod_{k=0}^{c-b-1} Z_{a+k, b+k}$ as $\tilde{Y}_{c-b, a+c, b-a}$,
 we get a tropical A'_r Y-system:

$$\tilde{Y}_{i, j-1, k} \tilde{Y}_{i, j+1, k} = \tilde{Y}_{i-1, j, k} \tilde{Y}_{i+1, j, k}$$

Theorem. $T(r) = \rho(T(r))$

Proof: substitute in $\vec{W}(r) = \overleftarrow{W}(r)$ the Sergeev–Kashaev–Volkov solution of the TE:

$$R_{abc} \rightarrow F_{abc}$$

$$R_{abc} \rightarrow F_{abc} \cdot [x_{ab}x_{bc}^{-1}y_{bc}^{-2}]_q$$

on the space of formal series in $r(r-1)/2$ variables x_{ab} , $1 \leq a < b \leq r$.

$$x_{ab} : x_{ab}^m \rightarrow x_{ab}^{m+1}, \quad y_{ab} : x_{ab}^m \rightarrow q^m x_{ab}^m$$

$$F_{abc} : x_{ab}^m x_{ac}^n x_{bc}^k \rightarrow x_{ab}^m x_{ac}^{k+m} x_{bc}^{n-m}$$

$F^2 = id$. F is an analogue of P in $R = P\check{R}$

$T(r) = \rho(T(r))$ for $r = 4$:

$$\begin{aligned}
 & [Z_{12}]_q [Z_{12}Z_{23}]_q [Z_{13}]_q [Z_{23}]_q [Z_{12}Z_{23}Z_{34}]_q [Z_{13}Z_{24}]_q [Z_{23}Z_{34}]_q [Z_{14}]_q [Z_{24}]_q [Z_{34}]_q \\
 & \qquad \qquad \qquad = \\
 & [Z_{14}]_q [Z_{24}]_q [Z_{34}]_q [Z_{14}Z_{13}]_q [Z_{24}Z_{23}]_q [Z_{14}Z_{13}Z_{12}]_q [Z_{13}]_q [Z_{23}]_q [Z_{13}Z_{12}]_q [Z_{12}]_q
 \end{aligned}$$

Reduction $Z_{14} = 0, Z_{34} = 0$ yields (A_2, A_2) identity ($6 \sim 6$):

$$\begin{aligned}
 & [Z_{12}]_q [Z_{12}Z_{23}]_q [Z_{13}]_q [Z_{23}]_q [Z_{13}Z_{24}]_q [Z_{24}]_q \\
 & = [Z_{24}]_q [Z_{24}Z_{23}]_q [Z_{13}]_q [Z_{23}]_q [Z_{13}Z_{12}]_q [Z_{12}]_q
 \end{aligned}$$

Reduction $Z_{23} = 0, Z_{24} = 0, Z_{34} = 0$ yields (A_3, A_1) identity ($3 \sim 6$):

$$[Z_{12}]_q [Z_{13}]_q [Z_{14}]_q = [Z_{14}]_q [Z_{14}Z_{13}]_q [Z_{14}Z_{13}Z_{12}]_q [Z_{13}]_q [Z_{13}Z_{12}]_q [Z_{12}]_q$$

$$(A_r, A_r) \longrightarrow (A'_r) \longrightarrow (A_{r_1}, A_{r_2}), \quad r_1 + r_2 = r$$