AGT correspondence, Bases in 2d CFT and Uglov polynomials

A. Belavin

Introduction

- The talk based on the joint work with M. Bershtein and G. Tarnopolsky .
- This work is a sequel to "Instanton moduli spaces and bases in coset conformal field theory"
- Alday, Gaiotto and Tachikawa (2009) proposed correspondence

  $\mathcal{N}=2$ SUSY $4d$ Gauge field theories $\longleftrightarrow$ $2d$ Conformal field theories
- In this work we study Conformal field theory using AGT conjecture for motivation.

CFT side.

- In QFT we have the complete set of the local fields $\{\mathcal{O}_k(\xi)\}$
$$\mathcal{O}_i(\xi)\mathcal{O}_j(0) = \sum_k C_{ij}^k(\xi)\mathcal{O}_k(0).$$
- In CFT the set $\{\mathcal{O}_k(\xi)\}$ can be decomposed as
$$\{\mathcal{O}_k(\xi)\} = \sum [\Phi_n].$$
The ancestor of each family Φ_n is called primary field . Other representatives of $[\Phi_n]$ are called descendant fields
- In two dimensions the conformal group is Vir $\otimes$ $\bar{\text{V}}$ir
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0},$$
- And hence the conformal family is a tensor product $[\Phi_n] = \pi_n \otimes \bar{\pi}_n$
$$[\Phi] = \{\Phi, \Phi^{(1)}, \Phi^{(1,1)}, \Phi^{(2)}, \dots\} \otimes \{\dots\},$$
where $\quad \Phi^{(n_1,n_2,\dots,n_k)} = L_{-n_1}L_{-n_2}\cdots L_{-n_k}\Phi$
- One can show that OPE of primary fields has a form
$$\Phi_1\Phi_2 = \sum\nolimits_k C_{12}^k\left(\Phi_k + \beta_1\Phi_k^{(1)} + \beta_{1,1}\Phi_k^{(1,1)} + \beta_2\Phi_k^{(2)} + \dots\right) \otimes (\dots)$$

CFT side.

Conformal blocks — holomorphic part of correlation function

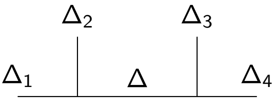
$$\langle \Phi_1(z_1,\bar{z}_1)\ldots\Phi_n(z_n,\bar{z}_n)\rangle.$$

It is convenient to choose $z_1=0$, $z_{n-1}=1$, $z_n=\infty$
Four point conformal block:

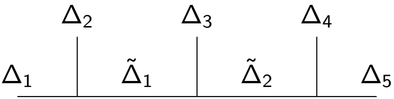
$$F\left(\begin{array}{ccc|c} \Delta_1 & \Delta_3 & \Delta & \multirow{2}{*}{q} \\ \Delta_2 & \Delta_4 & c & \end{array}\right)=\sum_{|\lambda|=|\mu|=n} q^n\langle\Phi_1|\Phi_2|\Phi_\Delta^{(\lambda)}\rangle(\langle\Phi_\Delta^{(\lambda)}|\Phi_\Delta^{(\mu)}\rangle)^{-1}\langle\Phi_\Delta^{(\mu)}|\Phi_3|\Phi_4\rangle$$

Five point conformal block:

$$F\left(\begin{array}{cccc|c} \Delta_1 & \Delta_3 & \Delta_5 & \tilde{\Delta}_2 & \multirow{2}{*}{q} \\ \Delta_4 & \Delta_4 & \tilde{\Delta}_1 & c & \end{array}\right)=\sum_{|\lambda|=|\mu|=n,|\nu|=|\rho|=m} q_1^n q_2^m\langle\Phi_1|\Phi_2|\Phi_{\tilde{\Delta}_1}^{(\lambda)}\rangle(\langle\Phi_{\tilde{\Delta}_1}^{(\lambda)}|\Phi_{\tilde{\Delta}_1}^{(\mu)}\rangle)^{-1}$$
$$\langle\Phi_{\tilde{\Delta}_1}^{(\mu)}|\Phi_3|\Phi_{\tilde{\Delta}_1}^{(\nu)}\rangle(\langle\Phi_{\tilde{\Delta}_2}^{(\nu)}|\Phi_{\tilde{\Delta}_2}^{(\rho)}\rangle)^{-1}\langle\Phi_{\tilde{\Delta}_2}^{(\rho}|\Phi_4|\Phi_5\rangle$$



4 point



5 point

AGT reason

- The algebraic reason of AGT: there exists a natural action of the symmetry algebra $\mathcal{A}$ of conformal field theory on equivariant cohomologies of moduli space of instantons $\mathcal{M}$.
- *Basis* in the equivariant cohomology space can be labeled by the fixed points of the torus. Thus the geometrical construction gives some *special basis of states* in the highest weight representations $\pi_{\mathcal{A}}$ of the algebra $\mathcal{A}$.
- This basis is already remarkable just because of its geometrical origin and possesses many nice properties. In general cases the action of algebra $\mathcal{A}$ haven't constructed explicitly but using the geometrical intuition one can predict the properties of the basis quoted above.

Properties of the geometrical basis

- To every torus fixed point $p \in \mathcal{M}$ correspond basic vector $v_p \in \pi_{\mathcal{A}}$.
- Basis v_p is orthogonal and the norm of the vector v_p equals to the determinant of the vector field v in the tangent space of p. The last expression is also denoted by Z_{vec}^{-1} (contribution of the vector multiplet).
- Matrix elements of a geometrically defined vertex operators have completely factorized form. The last expressions are also denoted by Z_{bif} (contribution of the bifundamental multiplet).
- There is a commutative algebra which diagonalizes in the basis v_p. Geometrically this algebra arise from the multiplication on cohomology classes.
- From the CFT point of view this basis is a basis in the Algebra of local fields. Z_{bif} provides explicit and simple expression for the coefficients of OPE.

Symmetry algebra

- The first question : What kind of the chiral symmetry in 2d CFT corresponds to a given $N = 2$ SUSY gauge theory ?

- AGT discovered that for $U(2)$ gauge theory on $\mathbb{R}^4$ it is CFT with the chiral algebra $\mathcal{A} = \mathcal{H} \oplus \text{Vir}$, where Vir is the Virasoro algebra and $\mathcal{H}$ is the Heisenberg algebra.

- Generalizing this V. Belavin, B. Feigin suggested that $U(r)$ gauge theory on $\mathbb{R}^4/\mathbb{Z}_p$ corresponds to CFT whose symmetry algebra is

$$\mathcal{A}(r,p) \overset{\text{def}}{=} \frac{\widehat{\mathfrak{gl}}(n)_r}{\widehat{\mathfrak{gl}}(n-p)_r} = \mathcal{H} \times \widehat{\mathfrak{sl}}(p)_r \times \frac{\widehat{\mathfrak{sl}}(r)_p \times \widehat{\mathfrak{sl}}(r)_{n-p}}{\widehat{\mathfrak{sl}}(r)_n},$$

Special cases and History.

- $r = 1,\ p = 1$. Algebra $\mathcal{A}(1,1) = \mathcal{H}$. The geometric action was constructed by Nakajima. The fixed points basis is given by Jack polynomials.
- $r = 2,\ p = 1$. Algebra $\mathcal{A}(2,1) = \mathcal{H} \oplus \text{Vir}$, where Vir is the Virasoro algebra. Initial AGT case. The construction of the basis is given by Alba, Fateev, Litvinov, Tarnopolsky.
- $r = 1,\ p = 2$. Algebra $\mathcal{A}(1,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$.
  This algebra has an subalgebra $\mathcal{A} = \mathcal{H} \oplus \mathcal{H}$
- $r = 2,\ p = 2$. Algebra $\mathcal{A}(2,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \text{NSR}$. There exist *different bases*. One was constructed in [BBFLT, 2011]

$p = 1$. Recollection of Instanton Moduli space

We denote Moduli space of Instantons in $U(r)$ gauge theory on $\mathbb{R}^4/Z_p$ as $\mathcal{M}(r,p)$. This space can be obtained from $\mathcal{M}(r,1)$ as its Z_p - invariant subspace. ADHM description of $\mathcal{M}(r,1)$:

$$\mathcal{M}(r,1) \cong \left\{ \begin{pmatrix} B_1, \\ B_2, \\ I, \\ J \end{pmatrix} \middle| \begin{array}{ll} \text{(i)} & [B_1, B_2] + IJ = 0 \\ \text{(ii)} & \begin{array}{l} \text{there are } N \text{ linear independent} \\ \text{vectors obtained by the action of} \\ \text{algebra genareted by } B_1 \text{ and } B_2 \text{ on} \\ I_1, I_2, \ldots, I_r \end{array} \end{array} \right\} \Big/ \mathrm{GL_N},$$

- N — is a topological number.
- B_j, I and J are $N \times N$, $N \times r$ and $r \times N$ complex matrices.
- $I_1, \ldots, I_r$ denote the columns of the matrix I.
- The GL_N action is given by

$$g \cdot (B_1, B_2, I, J) = (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}),$$

for $g \in \mathrm{GL}_N$.

$p = 1$. Torus action on the moduli space

- Abelian group (torus) $T = (U(1))^2 \times (U(1))^r$ acts on $\mathcal{M}(r, N)$.

  $$B_1 \mapsto t_1 B_1; \quad B_1 \mapsto t_1 B_1; \quad I \mapsto It; \quad J \mapsto t_1 t_2 t^{-1} J,$$

  infinitesimally this action is generated by vector field $v = (\epsilon_1, \epsilon_2, a) \in Lie(T)$ where $t_1 \equiv \exp \epsilon_1 \tau$, $t_2 \equiv \exp \epsilon_2 \tau$ and $t_v = \exp a\tau$.
- Points $p_{\vec{Y}}$ fixed under the torus action are labeled by the r-tuples of Young diagrams $\vec{Y} = (Y_1, \ldots, Y_r)$.

  The simplest cases $r = 1$ and $r = 2$.
- the Nekrasov partition function for pure $U(r)$ gauge theory (without matter) can be evaluated as an integral over the $\mathcal{M}(r, N)$ and equals the sum of fixed points contributions

$$Z_{\text{pure}}(\vec{a}, \epsilon_1, \epsilon_2 | q) = \sum_{k=0}^{\infty} \sum_{|\vec{Y}|=k} Z_{\text{vec}}(\vec{a}, \vec{Y} | \epsilon_1, \epsilon_2)\, q^k,$$

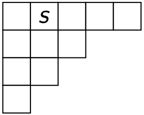
$p = 1$ fixed points contributions

Determinant of v on the tangent space of $p_{\vec{Y}}$

$$\det v\Big|_{p_{\vec{Y}}} = \prod_{i,j=1}^{r} \prod_{s\in Y_i} E_{Y_i,Y_j}(a_i - a_j|s)\big(\epsilon_1 + \epsilon_2 - E_{Y_i,Y_j}(a_i - a_j|s)\big),$$

where $E_{Y,W}(x|s) = x - \epsilon_1 \, \mathrm{l}_W(s) + \epsilon_2(\mathrm{a}_Y(s) + 1)$.

$\mathrm{a}_Y(s)$ and $\mathrm{l}_W(s)$ are correspondingly the arm length of the box s in the partition Y and the leg length in partition W



$$Z_{\text{vec}}(\vec{a}, \vec{Y}|\epsilon_1, \epsilon_2) \overset{\text{def}}{=} \prod_{i,j=1}^{r} \prod_{s\in Y_i} \Big(E_{Y_i,Y_j}(a_i - a_j|s)\big(\epsilon_1 + \epsilon_2 - E_{Y_i,Y_j}(a_i - a_j|s)\big)$$

$$Z_{\text{bif}}^{(r)}(m; \vec{a}', \vec{W}; \vec{a}, \vec{Y}|\epsilon_1, \epsilon_2) = \prod_{i,j=1}^{r} \prod_{s\in Y_i} \Big(\epsilon_1 + \epsilon_2 - E_{Y_i,W_j}(a_i - a'_j|s) - m\Big)$$

$$\prod_{t\in W_j} \Big(E_{W_j,Y_i}(a'_j - a_i|t) - m\Big),$$

$r = 1$, $p = 1$ case.

- In this case $\mathcal{A}$ is Heisenberg algebra with generators a_k
$$[\mathrm{a}_n, \mathrm{a}_m] = n\,\delta_{n+m,0}.$$
- The highest weight representation of this algebra (Fock module) is defined by the vacuum state $|0\rangle$
$$\mathrm{a}_n|0\rangle = 0 \quad \text{for} \quad n > 0,$$
and spanned by the vectors of the form
$$\mathrm{a}_{-k_1}\ldots\mathrm{a}_{-k_n}|0\rangle, \qquad k_1 \geq k_2 \geq \cdots \geq k_n.$$
- One can define another basis
$$|Y\rangle \overset{\text{def}}{=} \mathbf{J}_Y^{(\alpha)}(x)|0\rangle$$
where $\mathbf{J}_Y^{(1/g)}(x)$ is the Jack polynomial associated to the partition Y. $\alpha = -b^{-2}$ and the following identification is made
$$\mathrm{a}_{-k} = -ib\sum_j x_j^k.$$
- Examples:
$|\{1\}\rangle = a_{-1}|0\rangle; \quad |\{2\}\rangle = a_{-1}^2 + \frac{b^2}{1+b^2}a_{-2}; \quad |\{1,1\}\rangle = a_{-1}^2 - \frac{1}{2}a_{-2}$

$p=1$, $r=1$ case.

- System of Integrals of Motion $\mathbf{I}_k$ which acts diagonally in Jack basis

$$\mathbf{I}_1=\sum\nolimits_{k>0}\mathsf{a}_{-k}\mathsf{a}_k,$$

$$\mathbf{I}_2=iQ\sum\nolimits_{k>0}k\mathsf{a}_{-k}\mathsf{a}_k+\frac{1}{3}\sum_{i+j+k=0}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k.$$

- In terms of x_k

$$\mathbf{I}_1=\sum_{k>0}kx_k\frac{\partial}{\partial x_k},$$

$$\mathbf{I}_2=\frac{\alpha}{2}\sum\nolimits_{k>0}x_k^2\frac{\partial}{\partial x_k}+\sum\nolimits_{k\neq l}\frac{x_k^2}{x_k-x_l}\frac{\partial}{\partial x_k}.$$

- Vertex operator (Carlsson, Okounkov)

$$\mathrm{V}_\alpha=e^{(\alpha-Q)\varphi_-(1)}e^{\alpha\varphi_+(1)},$$

- The matrix elements has factorized form

$$\langle W|Y\rangle=\delta_{Y,W}Z_{\mathrm{vec}}(Y|b,b^{-1})$$

$$\langle W|\mathrm{V}_\alpha|Y\rangle=Z_{\mathrm{bif}}(\alpha;\vec{W};\vec{Y}|b,b^{-1})$$

$r = 1, p = 2$. Algebraic (Homogeneous) construction

- Algebra $\mathcal{A}(1,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$.
  This algebra has an subalgebra $\mathcal{A} = \mathcal{H} \oplus \mathcal{H}$
- Algebra $\mathcal{A}(1,2)$ has a basis w_n for $\mathcal{H}$ and f_n, e_n, h_n for $\widehat{\mathfrak{sl}}(2)_1$
- Consider Homogeneous (Frenkel-Kac) construction of the representation of $\widehat{\mathfrak{sl}}(2)_1$ based on Heisenberg algebra $\langle h_n \rangle$.
- Denote by $\mathrm{F_P}$ the Fock representation of Heisenberg algebra $\langle h_n \rangle$ with vacuum vector v_P:

$$h_n v_P = 0 \text{ for } n > 0 \quad h_0 v_P = P v_P$$

- Denote by D, $D\colon \mathrm{F}_P \to \mathrm{F}_{P+1}$ operator defined by commutation relations

$$[D, h_n] = 0 \text{ for } n \neq 0 \quad [h_0, D] = D$$

  Each integrable representation of $\widehat{\mathfrak{sl}}(2)_1$ has two natural grading. The first of them is a h_0 grading. Another grading is defined by Sugawara operator $L_0^{\widehat{\mathfrak{sl}}}$
- Then Character of representation $\Pi_{h,k}$ defined by formula

$$\chi^{h,k}_{\widehat{\mathfrak{sl}}(2)_1} = \operatorname{Tr} q^{L_0^{\widehat{\mathfrak{sl}}}} t^{h_0/2}\Big|_{\Pi_{h,k}}.$$

$r=1, p=2$. Algebraic (Homogeneous) construction

- The level 1 $\widehat{\mathfrak{sl}}(2)_1$ representations $\Pi_{0,1}$ and $\Pi_{1,1}$ can be realized in terms of one Heisenberg algebra h_n as sums of Fock modules.
- Namely $\Pi_{0,1}=\bigoplus_{n\in\mathbb{Z}}\mathrm{F}_{2n}, \qquad \Pi_{1,1}=\bigoplus_{n\in\mathbb{Z}}\mathrm{F}_{2n+1}$
- The generators e_i and f_i are defined in both representations by the same formulae

$$e(z)=\sum_{n\in\mathbb{Z}}e_nz^{-n}=D^2:\exp\left(-2\sum_{n\in\mathbb{Z}}\frac{h_n}{2n}z^{-n}\right):,$$

$$f(z)=\sum_{n\in\mathbb{Z}}f_nz^{-n}=D^{-2}:\exp\left(2\sum_{n\in\mathbb{Z}}\frac{h_n}{2n}z^{-n}\right):.$$

- Characters of these representations follow from this construction:

$$\chi^{0,1}_{\widehat{\mathfrak{sl}}(2)_1}=\sum_{n\in\mathbb{Z}}t^nq^{n^2}\chi_B(q),\quad \chi^{1,1}_{\widehat{\mathfrak{sl}}(2)_1}=\sum_{r-\frac{1}{2}\in\mathbb{Z}}t^rq^{r^2}\chi_B(q),$$

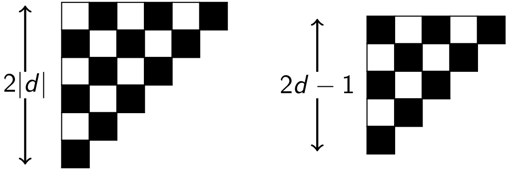
  where $\quad \chi_B(q)=\prod_{k\in\mathbb{Z}_{>0}}\frac{1}{1-q^k}$
- Since our algebra is $H\oplus\widehat{\mathfrak{sl}}(2)_1$, we will consider the tensor products $\mathcal{L}_{0,1}=\bigoplus_{n\in\mathbb{Z}}\mathcal{F}\otimes\mathrm{F}_{2n}, \qquad \mathcal{L}_{1,1}=\bigoplus_{n\in\mathbb{Z}}\mathcal{F}\otimes\mathrm{F}_{2n+1}$ where $\mathcal{F}$ is a Fock representation of $\mathcal{H}$.

$r = 1, p = 2$. Comparison of Algebraic and Geometric data

- We want to construct the special basis in the representation of algebra $\mathcal{A}(1,2) = H \oplus \widehat{\mathfrak{sl}}(2)_1$. The basic vectors labeled by torus fixed points on the moduli space $\mathcal{M}(1,2)$.
- In this case torus fixed points are labeled by two colors colored Young diagrams λ with angle colored in the color σ . We will denote such basis by J_{λ^σ}. The color of the angle σ corresponds to the highest weight of representations $\mathcal{L}_{\sigma,1}$.
- To check our proposal we compare the dimensions of the graded subspaces of $\mathcal{L}_{\sigma,1}$ and of the graded subspaces of the fixed points in Instanton moduli space for case $p = 2, r = 1$ which are labeled by Young diagrams colored in two colors .
- The box $s \in \lambda$ with coordinates (i,j) has color $i - j + \sigma$ and $\sigma = 0, 1$ is a color of the angle.

$r = 1$, $p = 2$. Comparison of Algebraic and Geometric data

- We will denote such Young diagram by λ^σ in order to stress the coloring. By $N_i(\lambda^\sigma)$ we denote the number of boxes of color i, $i = 0, 1$. Let $d(\lambda^\sigma) = N_0(\lambda^\sigma) - N_1(\lambda^\sigma)$.
- There exists a smallest partition μ with $d(\mu) = d$ consist of $2d^2 - d$ boxes and has a “triangular” form with edge length $2|d|$ for $d \leq 0$ and $2d - 1$ for $d > 0$. Such minimal diagram is called 2-core . For any partition λ its 2-core is denoted by $\widetilde{\lambda}$



- Lemma. The number of partitions λ such that $d(\lambda) = d$ and $|\lambda| - |\widetilde{\lambda}| = 2n$ equals to the number of pairs of partitions (μ_1, μ_2) such that $|\mu_1| + |\mu_2| = n$.

$r=1$, $p=2$. Comparison of Algebraic and Geometric data

- In terms of generating functions for colored partitions with fixed d $\chi^{(1)}_{d,\sigma}(q)=\sum_{\lambda^\sigma,d(\lambda^\sigma)=d} q^{|\lambda|/2}=q^{\frac{2d^2-(-1)^\sigma d}{2}}\chi_B^2$
- Using this formula we get

$$\chi^{(1)}_\sigma=\sum_{\lambda^\sigma} q^{\frac{|\lambda|+d(\lambda^\sigma)+\sigma/2}{2}}t^{d(\lambda^\sigma)+\sigma/2}=\sum_d (t\sqrt{q})^{d+\sigma/2}\chi^{(1)}_{d,\sigma}(q)=$$

$$=\sum_d (t\sqrt{q})^{d+\sigma/2}q^{\frac{2d^2-(-1)^\sigma d}{2}}\chi_B(q)^2=\sum_{n\in\mathbb{Z}+\frac{\sigma}{2}} t^n q^{n^2}\chi_B(q)^2$$

- So these expressions coincide with characters of algebra $\mathcal{A}(1,2)$ if we identify geometrical and algebraic data as follows:

$$h_0(J_{\lambda^\sigma})=2d(\lambda^\sigma)+\sigma,\qquad L_0(J_{\lambda^\sigma})=(2|\lambda|+h_0)/4,$$

where L_0 is a total degree with respect to algebra $H\oplus\widehat{\mathfrak{sl}}(2)_1$:

$$L_0=L_0^{\widehat{\mathfrak{sl}}(2)_1}+L_0^H=L_0^{\widehat{\mathfrak{sl}}}+\sum_{k\in\mathbb{Z}_{>0}} w_{-k}w_k$$

$r = 1, p = 2$ case. Pricipal realization

- To find the geometrical basis we use another realization of $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$
- The generators of $\widehat{\mathfrak{sl}}(2)_1$ can be written in terms new Heisenberg generators a_n as
  $e_n = \frac{1}{2}(a_{2n+1} - b_{2n+1}) \quad f_n = \frac{1}{2}(a_{2n+1} + b_{2n+1}) \quad h_n = b_{2n} + \frac{1}{2}\delta_{n,0}$
- where

$$\sum_n b_n z^{-n} = b(z) = \frac{(-1)^{\sigma+1}}{2} \exp\left(2\sum_n \frac{a_{2n+1}}{-2n-1} z^{-2n-1}\right)$$

- Elements of the additional Heisenberg algebra $\mathcal{H}$

$$w_n = a_{2n}$$

- Indeed this realization of $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$ arises from Level One realization of Ding-Iohara algebra given by Awata,Feigin,Hoshino,Kanai,Shiraishi,Yanagida (AFHKSY)

The deformed (Ding-Iohara) algebra

- The Ding-Iohara algebra generated by Drinfeld currents $E(z), F(z), \psi^{+}(z), \psi^{-}(z)$. The level 1 realisation of this algebra is given in Fock representation on (deformed) Heisenberg algebra.
- Deformed Heisenberg

  $$[a_n, a_m] = n\frac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}.$$

  Fock representation defined by vacuum $|0\rangle$: $a_n|0\rangle = 0$ for $n > 0$.
- Realization:

  $$E(z) := u\exp\Big(\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n}a_{-n}z^{n}\Big)\exp\Big(-\sum_{n=1}^{\infty}\frac{1-t^{n}}{n}a_{n}z^{-n}\Big),$$

  $$F(z) := \frac{1}{u}\exp\Big(-\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n}(\frac{t}{q})^{n/2}a_{-n}z^{n}\Big)\exp\Big(\sum_{n=1}^{\infty}\frac{1-t^{n}}{n}(\frac{t}{q})^{n/2}a_{n}z^{-n}\Big)$$

  $$\psi^{+}(z) := \exp\Big(-\sum\nolimits_{n=1}^{\infty}\frac{1-t^{n}}{n}(1-t^{n}q^{-n})(\frac{t}{q})^{-n/4}a_{n}z^{-n}\Big),$$

  $$\psi^{+}(z) := \exp\Big(\sum\nolimits_{n=1}^{\infty}\frac{1-t^{-n}}{n}(1-t^{n}q^{-n})(\frac{t}{q})^{-n/4}a_{-n}z^{n}\Big).$$

The deformed (Ding-Iohara) algebra

- Geometrical basis: $J_\lambda(q,t)$ (Macdonald polynomials).

$$J_\varnothing = |u\rangle, \qquad J_{(1)} = (1-t)a_{-1}|u\rangle,$$

$$J_{(2)} = \left(\frac{1}{2}(1+q)(1-t)^2 a_{-1}^2 + \frac{1}{2}(1-q)(1-t^2)a_{-2}\right)|u\rangle,$$

$$J_{(1,1)} = \left(\frac{1}{2}(1-t)^2(1+t)(a_{-1}^2 - a_{-2})\right)|u\rangle.$$

- Vertex operator $\Phi(z)\colon \mathcal{F}_u \to \mathcal{F}_v$ has form

$$\Phi(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{v^n - (t/q)^n u^n}{1-q^n} \cdot \frac{a_{-n} z^n}{n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{v^{-n} - u^{-n}}{1-q^{-n}} \cdot \frac{a_n z^{-n}}{n}$$

- Then matrix element of vertex operator has the form

$$\langle J_\lambda(q,t)|\Phi(z)|J_\mu(q,t)\rangle = N_{\lambda,\mu}\left(\frac{qv}{tu}\right) \cdot \left(\frac{tu}{q}\right)^{|\lambda|} \left(-\frac{v}{q}\right)^{-|\mu|} t^{n(\lambda)} q^{n(\mu')} z^{|\lambda|-|\mu|},$$

$$N_{\lambda,\mu}(u) = \prod_{s\in\lambda}(1 - uq^{-a_\mu(s)-1}t^{-l_\lambda(s)}) \cdot \prod_{t\in\mu}(1 - uq^{a_\lambda(t)}t^{l_\mu(t)+1}).$$

The limit of the Ding-Iohara algebra

- Take the following limit

$$q = \omega_p e^{-\tau\epsilon_2}, \quad t = \omega_p e^{\tau\epsilon_1}, \qquad \omega_p = e^{i\frac{2\pi}{p}}, \quad \tau \to 0,$$

- In the classical case $p = 1$ we have $\omega_p = 1$ and Macdonald polynomials $J_\lambda^{q,t}$ convert to Jack polynomial $J_\lambda^{(-1/b^2,1)}$.
- $p = 2$ case we have $\omega_p = -1$ and $q = -e^{-\tau\epsilon_2}$, $t = -e^{\tau\epsilon_1}$, $\tau \to 0$. The limit of Heisenberg algebra reads

$$[a_n, a_m] = \begin{cases} -nb^{-2}\delta_{n+m,0}, & \text{if} \quad n \equiv 0 \bmod 2 \\ n\delta_{n+m,0}, & \text{if} \quad n \equiv 1 \bmod 2 \end{cases}$$

  We see that Heisenberg algebra fall into pieces: even and odd part.
- The limit of Fock representation of Ding-Iohara algebra coincides with Principal representation of $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$

basis

- The limit of matrix elements of the vertex operator coinside with Z_{bif}
- Macdonald polynomials $J_\lambda^{q,t}$ convert to rank 2 Uglov polynomials $J_\lambda^{(-1/b^2,2)}$
- First examples has the form:

$$J_\varnothing^{(2)} = |\kappa\rangle_q, \qquad J_{(1)}^{(2)} = a_{-1}|\kappa\rangle_q, \qquad J_{(2)}^{(2)} = (b^{-1}a_{-1}^2 - ia_{-2})|\kappa\rangle_q,$$

$$J_{(1,1)}^{(2)} = (ba_{-1}^2 - ia_{-2})|\kappa\rangle_q, \qquad J_{(3)}^{(2)} = (\frac{1}{3b}a_{-1}^3 - ia_{-2}a_{-1} + \frac{2}{3b}a_{-3})|\kappa\rangle_q,$$

$$J_{(2,1)}^{(2)} = -\frac{1}{3}(a_{-1}^3 - a_{-3})|\kappa\rangle_q, \qquad J_{(1,1,1)}^{(2)} = (\frac{b}{3}a_{-1}^3 - ia_{-2}a_{-1} + \frac{2b}{3}a_{-3})|\kappa\rangle_q$$

- In the limit we find system of Integrals of Motion which acts diagonally the basis $J_\lambda^{(2)}$. For example

$$e_0 \to 1 - 2h_0, \qquad h_0(J_\lambda^{(2)}|\kappa\rangle_q) = (2d(\lambda) + q)\, J_\lambda^{(2)}|\kappa\rangle_q$$

Principal and Homogeneous realizations

Their connection is as follows

- $\widehat{\mathfrak{sl}}(2)_1$ generators

$$a_{2n+1} = f_{n+1} + e_n, \quad b_{2n+1} = f_{n+1} - e_n, \quad b_{2n} = h_n - \frac{1}{2}\delta_{n,0}$$

- Generators h and w from a

$$\sum_n b_n z^{-n} = b(z) = \frac{(-1)^{q+1}}{2} \exp\left(2 \sum_n \frac{a_{2n+1}}{-2n-1} z^{-2n-1} \right)$$

$$w_n = a_{2n}$$

- The basis in h, w

$$J^{(2)}_{\varnothing} = |\kappa\rangle_0, \quad J^{(2)}_{(2)} = -(iw_{-1} + b^{-1}h_{-1})J^{(2)}_{\varnothing}, \quad J^{(2)}_{(1,1)} = -(iw_{-1} + bh_{-1})J^{(2)}_{\varnothing}$$

$$J^{(2)}_{(1,1,1,1)} = (-2ibw_{-2} - w^2_{-1} + 2ibw_{-1}h_{-1} + bh^2_{-1} - 2bh_{-2})J^{(2)}_{\varnothing},$$

$$J^{(2)}_{(2,1,1)} = \left(-2ibw_{-2} - w^2_{-1} + i(b^{-1} - b)w_{-1}h_{-1} + b^2h^2_{-1} - (1-b^2)h_{-2}\right) J$$

$$J^{(2)}_{(2,2)} = \left(-i(b + b^{-1})w_{-2} - w^2_{-1} + h^2_{-1}\right) J^{(2)}_{\varnothing}.$$

Such formulae much simpler then in terms of a_n

r=2 ,p=2 case, Algebras and representations

- We want to construct the special (geometric) basis for the algebra $A(2,2) = H \oplus sl(2)_2 \oplus \mathrm{NSR}$.
- We consider only the NeveuSchwarz sector of the NSR algebra i.e. the algebra generated by $L_n, G_r,\ n \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$ with relations:

$$
\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c_{\mathrm{NSR}}}{8}(n^3 - n)\delta_{n+m},\\
&\{G_r, G_s\} = 2L_{r+s} + \frac{1}{2}c_{\mathrm{NSR}}(r^2 - \frac{1}{4})\delta_{r+s,0},\\
&[L_n, G_r] = \left(\frac{1}{2}n - r\right) G_{n+r}.
\end{aligned}
$$

- The representations of algebra $\mathcal{A}(2,2) = H \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \mathrm{NSR}$ is a tensor product of representations of the algebras H, $\widehat{\mathfrak{sl}}(2)_2$ and NSR. For the NSR we take the Verma module with highest weight parametrized by the P. The highest weight vector is denoted by $|P\rangle$

$$
L_n|P\rangle = G_r|P\rangle = 0 \quad \text{for} \quad n, r > 0, \qquad L_0|P\rangle = \Delta_{\mathrm{NS}}|P\rangle,
$$

The character of this representation $\chi_{\mathrm{NSR}}(q) = \chi_B(q)\chi_F(q)$.

$r = 2$, $p = 2$ case, Algebras and representations

- The $\widehat{\mathfrak{sl}}(2)$ algebra has three integrable representations of level 2: $\Pi_{0,2}$, $\Pi_{1,2}$ and $\Pi_{2,2}$. In the Neveu–Schwarz we take the direct sum $\Pi_{0,2} \oplus \Pi_{2,2}$. Its character equals:

$$\chi^{(0,2)}_{\widehat{\mathfrak{sl}}}(q,t) + \chi^{(0,2)}_{\widehat{\mathfrak{sl}}}(q,t) = \prod_{r+\frac{1}{2}\in\mathbb{Z}_{>0}} (1 + \frac{q^r}{t})(1+q^r)(1+tq^r) = \sum_{n\in\mathbb{Z}} t^n q^{n^2/2} \chi_B \chi_F$$

- We want to construct a special basis in the representation of algebra $\mathcal{A}(2,2)$ with properties from the Introduction. The basic vectors correspond to fixed points on the moduli space $\mathcal{M}(2,2)$. In this case torus fixed points are labelled by pairs of two colors colored Young diagrams λ_1, λ_2 with angles colored in the same color σ. We will denote such basis by $J_{\lambda_1^\sigma,\lambda_2^\sigma}$ or $J_{\vec{\lambda}^\sigma}$ for short. As in $r = 1$ case such basis respects grading:

$$h_0(J_{\vec{\lambda}^\sigma}) = 2d(\lambda_1^\sigma) + 2d(\lambda_2^\sigma) + 2\sigma, \qquad L_0(J_{\vec{\lambda}^\sigma}) = (2|\lambda| + h_0)/4$$

- The existence of such basis means that we can compute the character of representation of algebra $\mathcal{A}(2,2)$. Using the formula for $\chi^{(1)}$ we qet:

$$\chi^{(2)}(q,t)=\sum_{\sigma=0,1}\sum_{\lambda_1^\sigma,\lambda_2^\sigma} q^{\frac{|\vec{\lambda}|+d(\vec{\lambda})+\sigma)}{2}}t^{d(\vec{\lambda})+\sigma}=\sum_{\sigma,d_1,d_2}\chi^{(1)}_{\sigma,d_1}(q)\chi^{(1)}_{\sigma,d_2}(q)(t\sqrt{q})^{d_1+d_2+\sigma}=$$

$$\chi_B(q)^4\sum_{d_1,d_2} q^{\frac{2d_1^2+2d_2^2-d_1-d_2}{2}}(t\sqrt{q})^{d_1+d_2}+q^{\frac{2d_1^2+2d_2^2+d_1+d_2}{2}}(t\sqrt{q})^{d_1+d_2+1}=$$

$$\chi_B(q)^4\sum_{d}(t\sqrt{q})^d\sum_{d_1}\left(q^{\frac{(2d_1-d)^2+d^2-d}{2}}+q^{\frac{(2d_1-d+1)^2+d^2-d}{2}}\right)=\chi_B(q)^4\sum_{d,k}q^{\frac{d^2+k^2}{2}}t^d$$

Using the Jacobi triple product identity this expression can be rewritten as

$$\chi_B(q)^4\sum_{d,k}q^{\frac{d^2+k^2}{2}}t^d=\chi_B(q)^2\prod_{r+\frac{1}{2}\in\mathbb{Z}_{>0}}(1+\frac{q^r}{t})(1+q^r)^2(1+tq^r)=$$

$$\chi_B(q)\cdot\left(\chi^{(0,2)}_{\widehat{\mathfrak{sl}}}(q,t)+\chi^{(0,2)}_{\widehat{\mathfrak{sl}}}(q,t)\right)\cdot\chi_F(q)\chi_B(q)=\chi_{H\oplus\widehat{\mathfrak{sl}}(2)_2\oplus\mathrm{NSR}}.$$

where we used the formula for $\left(\chi^{(0,2)}_{\widehat{\mathfrak{sl}}}(q,t)+\chi^{(0,2)}_{\widehat{\mathfrak{sl}}}(q,t)\right)$

Free field realization

- The generators of $\mathcal{H}$ are denoted by w_n as in $r = 1$ case. So the full set of generators of algebra $\mathcal{A}$ consist of $w_n; e_n, f_n, h_n; L_n, G_r$.
- We will use the free field realization of the NSR and $\widehat{\mathfrak{sl}}(2)_2$ algebras. The free realization means that we consider the representation of Heisenberg algebra and Majorana fermion algebra and define the action of the mentioned before algebras on this representation. This method is very useful since Heisenberg and Majorana fermion algebra are much simpler.
- The algebra $\widehat{\mathfrak{sl}}(2)_2$ realized by generators h_n, D, χ_r.

$$e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n} = \chi(z) D^2 : \exp\left(-2\sum_n \frac{h_n}{4n} z^{-n}\right) :$$

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n} = \chi(z) D^{-2} : \exp\left(2\sum_n \frac{h_n}{4n} z^{-n}\right) :$$

where $\chi(z) = \sum_r \chi_r z^{-r-1/2}$,

$$[h_n, h_m] = 4n\delta_{n+m,0}, \qquad \{\chi_r, \chi_s\} = \delta_{r+s,0}$$

Free field realization

For the *NSR* algebra we consider in the representation of algebra with generators c_n, ψ_r:

$$[c_n, c_m] = n\delta_{n+m,0}, \qquad \{\psi_r, \psi_s\} = \delta_{r+s,0}$$

The representation of NSR of momentum P defined by formulas:

$$L_n = \frac{1}{2}\sum_{k\neq 0,n} c_k c_{n-k} + \frac{1}{2}\sum_r (r - \frac{n}{2})\psi_{n-r}\psi_r + \frac{i}{2}(Qn - 2P)c_n$$

$$L_0 = \sum_{k>0} c_{-k}c_k + \sum_{r>0} r\psi_{-r}\psi_r + \frac{1}{2}(\frac{Q^2}{4} - P^2)$$

$$G_r = \sum_{n\neq 0} c_n\psi_{r-n} + i(Qr - P)\psi_r.$$

Hence in free realization of representation of algebra A we have generators $w_n; h_n, D, \chi_r; c_n, \psi_r$.

The limit of DI algebra

The level 2 representations of Ding-Iohara algebra realized in terms of two Heisenberg algebras $a_n^{(1)}$, $a_n^{(2)}$. The generators of $\mathcal{A}(2,2)$ realized by formulae:

$$w_n = a_{2n}^{(1)} + a_{2n}^{(2)}, \qquad c_n = \frac{a_{2n}^{(1)} - a_{2n}^{(2)}}{2}$$

$$\sum_n \frac{h_n}{z^{2n}} - 1 = \frac{(-1)^{\sigma+1}}{4}\Big(\exp(2\phi^{(1)}) + \exp(2\phi^{(2)}) + \exp(-2\phi^{(1)}) + \exp(-2\phi^{(2)})\Big)$$

$$\sum_r \frac{\chi_r}{z^{2r}} = \frac{i}{2\sqrt{2}}\Big(\exp(\phi^{(1)} + \phi^{(2)}) - \exp(-\phi^{(1)} - \phi^{(2)})\Big),$$

$$\sum_r \frac{\psi_r}{z^{2r}} = \frac{i}{2\sqrt{2}}\Big(\exp(\phi^{(1)} - \phi^{(2)}) - \exp(-\phi^{(1)} + \phi^{(2)})\Big),$$

where

$$\phi^{(1)} = \sum_n \frac{a_{2n+1}^{(1)}}{-2n-1} z^{-2n-1} \qquad \phi^{(2)} = \sum_n \frac{a_{2n+1}^{(2)}}{-2n-1} z^{-2n-1}.$$

The limit of DI algebra

- Vectors written in terms of Heisenberg algebras $a_n^{(1)}$ and $a_n^{(2)}$ can be rewritten in terms of Heisenberg algebras h_n^1, w_n^1 and h_n^2, w_n^2 as in $r = 1$ case

$$w_n^1 = a_{2n}^1 \qquad \sum h_n^1 z^{-2n} - \frac{1}{2} = \frac{(-1)^{\sigma+1}}{4}\left(\exp(2\phi^{(1)}) + \exp(-2\phi^{(1)})\right)$$
$$w_n^2 = a_{2n}^2 \qquad \sum h_n^2 z^{-2n} - \frac{1}{2} = \frac{(-1)^{\sigma+1}}{4}\left(\exp(2\phi^{(2)}) + \exp(-2\phi^{(2)})\right)$$

- In terms of $\mathcal{A}(2,2)$ bosonisation $w_n, h_n, \chi_r, c_n, \psi_r$

$$h_n^1 = \frac{1}{2}h_n - \sum \chi_r \psi_{n-r} \qquad h_n^2 = \frac{1}{2}h_n + \sum \chi_r \psi_{n-r}$$

The Basis

- In terms of $a^{(1)}, a^{(2)}$

$$J_{\varnothing,(1)} = a^{(1)}_{-1} \qquad J_{\varnothing,(1)} = -\frac{Q}{2P}a^{(1)}_{-1} + a^{(2)}_{-1}$$

$$J_{(2),\varnothing} = b^{-1}(a^{(1)}_{-1})^2 - ia^{(1)}_{-2} \qquad J_{(1,1),\varnothing} = b(a^{(1)}_{-1})^2 - ia^{(1)}_{-2})$$

$$J_{\varnothing,(2)} = b^{-1}(a^{(2)}_{-1})^2 - ia^{(2)}_{-2} - \frac{Q}{2P}\left(b^{-1}(a^{(1)}_{-1})^2 - ia^{(1)}_{-2}\right),$$

- In terms of $w; h, \chi; L, G$

$$J_{(1),\varnothing} = G_{-\frac{1}{2}} + \frac{(Q+2P)}{2}\chi_{-\frac{1}{2}}, \qquad P_{\varnothing,(1)} = G_{-\frac{1}{2}} + \frac{(Q-2P)}{2}\chi_{-\frac{1}{2}},$$

$$J_{(2),\varnothing} = L_{-1} + \frac{1}{b}\chi_{-\frac{1}{2}}G_{-\frac{1}{2}} + \frac{i(Q+2P)}{2}w_{-1} + \frac{Q+2P}{2b}h_{-1};$$

$$J_{(1,1),\varnothing} = L_{-1} + b\chi_{-\frac{1}{2}}G_{-\frac{1}{2}} + \frac{i(Q+2P)}{2}w_{-1} + \frac{b(Q+2P)}{2}h_{-1};$$

$$J_{\varnothing,(2)} = L_{-1} + \frac{1}{b}\chi_{-\frac{1}{2}}G_{-\frac{1}{2}} + \frac{i(Q-2P)}{2}w_{-1} + \frac{Q-2P}{2b}h_{-1}$$

Conclusion

Results.

- The general construction of the basis for $\mathcal{A} = \mathcal{A}(r, p)$ from the deformed algebra.
- For $r = 1$ this construction leads for the Uglov polynomials.
- For $r = 2$, $p = 2$ the formulas for basis J_{λ_1,λ_2}. In case $\lambda_2 = \varnothing$ this basis converts to the Uglov polynomials.

Open questions

- To describe vertex operators
- Find the corresponding integrable system
- Consider $(r, p) > (2, 2)$ cases in more detail