1 Abstract

We present a new approach to the static finite temperature correlation functions of the Heisenberg chain based on functional equations. An inhomogeneous generalization of the n-site density operator is considered. The lattice path integr- rational formulation with a finite but arbitrary Trotter number allows one to define a set of discrete functional equations with respect to the spectral parameters. These equations yield a unique characterization of the density operator. Our functional equations are a discrete version of the reduced q-Knizhnik-Zamolodchikov equations which played a central role in the study of the zero temperature case. As a natural result, and independent of the arguments given by Jin, Miu, and Smirnov (2009 J. Phys. A 42 304018) we prove that the inhomogeneous finite temperature correlation functions have the same remarkable structure as for zero temperature: they are a sum of products of nearest-neighbor correlators.

2 Hamiltonian and R-matrix

The Hamiltonian of the spin-1/2 Heisenberg chain is given by

\[ H = \sum_{n=1}^{N} \left[ J \sigma_n^x \sigma_{n+1}^x + J \sigma_n^y \sigma_{n+1}^y + J \sigma_n^z \sigma_{n+1}^z \right] \]

where \( J_x, J_y, \) and \( J_z \) are the Pauli matrices. We are interested in the thermodynamic limit of (2). The matrix of the classical counterpart of the Heisenberg chain, the six-vertex model, is given by

\[ R(\xi_1, \xi_2) = \frac{1}{1 + \xi_1 \xi_2} \left[ \begin{array}{cc} 1 & \xi_1 \\
-\xi_2 & 1 \end{array} \right] \]

for \( \Delta = (-\xi) \) the functions \( \xi \) and \( \eta \) are defined as

\[ \xi_n(\sigma, \rho, \lambda) = \frac{1}{2} \left( 1 - a_n(\sigma, \rho, \lambda) \right) \]

with \( a_n(\sigma, \rho, \lambda) = \sigma_\alpha \rho_\beta \lambda_{\alpha \beta} \).

The matrix elements of the R-matrix are considered as local Boltzmann weights of the six-vertex model and are graphically represented by vertices with spin variables \( \mu, \nu \) at the ends of the bonds and spectral parameters \( \lambda \) associated with the lines.

\[ R(\lambda_1, \lambda_2) = \sum_{\mu, \nu} R(\lambda_1, \mu) \xi(\sigma_\mu \rho_\nu) \lambda_{\nu (\sigma \rho, \lambda)} \]

\[ \xi(\sigma_\mu \rho_\nu) = \frac{1}{1 + \xi(\sigma_\mu, \rho_\nu)} \]

Figure 1: Graphical illustration of \( R(\lambda_1, \lambda_2) \).

3 The inhomogeneous n-site density operator for finite Trotter number

A very important object for the study of the model on the semi-infinite cylinder is the column-to-column transfer matrix (3). This matrix (the limit \( N \to \infty \)) is also known as the quantum transfer matrix of the Heisenberg chain. We define the corresponding inhomogeneous quantum monodromy matrix as

\[ T(\lambda) = R(\lambda_1, \lambda_2) \cdots R(\lambda_{n-1}, \lambda_n) R(\lambda_n, \lambda_1) \]

Figure 2: Graphical illustration of \( T(\lambda) \) for the quantum monodromy matrix.

4 The discrete functional equation

For zero temperature the density operator satisfies the functional equation

\[ \phi_{\lambda_1, \lambda_2} (p, q, \lambda) \phi_{\lambda_2, \lambda_1} (p, q, \lambda) = \phi_{\lambda_1, \lambda_2} (p, q, \lambda) \phi_{\lambda_2, \lambda_1} (p, q, \lambda) \]

(3a) for arbitrary complex \( \lambda \). Under suitable asymptotic conditions and analyticity the solution to this functional equation is unique [3]. At finite temperature resp. finite Trotter number the functional equation does not hold for arbitrary \( \lambda \). The proof of (3a) for \( T = 0 \) is based on so-called ‘2-imari’s arguments relating the partition functions of different lattices. At \( T = 0 \) terms disappear as they can be viewed as boundary terms at infinity. For finite Trotter number these terms matter. When we attempted to quantify these terms we were surprised to realize that the unwanted terms do not appear for the finite set of arguments \( \lambda = \{ \lambda_0 = \cdots = \lambda_N = \lambda \} \).

Interestingly, equation (3a) for (3b) and asymptotic and analytic properties of the density operator are sufficient to uniquely fix the solution [4].

5 The solution for \( D_{\lambda_1, \lambda_2} \)

The solution for the n-site density operator of the XXX model can be stated in terms of a single nearest neighbor correlator \( \omega \) in the form

\[ D_{\lambda_1, \lambda_2} = \sum \left( \prod \phi_{\lambda_1, \lambda_2} \phi_{\lambda_2, \lambda_1} \right) \omega^{\lambda_1, \lambda_2} \]

with respect to the normalized eigenvectors \( \sigma^x, \sigma^y, \sigma^z \) of the spin \( \frac{1}{2} \) in the state \( V \) the matrix is denoted as

\[ T(\lambda)^{\sigma_1 \sigma_2} \]

Figure 3: LHS: Using the crossing symmetry for the quantum monodromy matrix \( T(\lambda) \) acting on site \( n \) the direction of \( (\lambda_n - 1) \) is changed, thereby raising the spectral parameter by one. For the special case that \( \lambda_n \) is equal to one of the parameters \( \nu \) we choose \( \nu \) then we find the RHS by use of the initial condition and unitarity RHS: the two traces in eq. (4a) and (4b) contractions over the pair \( (\nu_1, \lambda_1) \) and \( (\nu_2, \lambda_2) \) respectively, yield the same object. In other words we obtain equation (3a) for (3b).

The solution for the XXZ model is obtained by the same algebraic construction of structure coefficients in terms of two independent nearest neighbor correlators [4].

6 The discrete functional equation

The information about \( \lambda_0 \) and \( \omega \) is obtained from the auxiliary function

\[ \ln(\lambda_0) = \ln(\omega) + \frac{1}{2} \int \frac{d\omega}{1 + \omega} \ln(1 + \omega) d\omega \]

Leading eigenvalue of the modified transfer matrix:

\[ \ln(\lambda_0) = \ln(\omega) + \frac{1}{2} \int \frac{d\omega}{1 + \omega} \ln(1 + \omega) d\omega \]

It follows for \( \lambda \):

\[ \lambda = \frac{1}{2} \int \frac{d\omega}{1 + \omega} \ln(1 + \omega) d\omega \]

With the function \( \lambda(\omega) = \frac{1}{2} \ln(1 + \omega) \) determined by the integral equation

\[ G(\omega, \lambda) = \frac{1}{2} \ln(1 + \lambda(\omega)) \]

The driving terms \( \omega_1, \omega_2 \) depend on the spectral parameters \( \nu_1, \nu_2, \nu_3 \). Specializing to the physical values (2) and performing the Trotter limit is straightforward.

1 Some first results for Temperley-Lieb systems

Spin chains of Temperley-Lieb type can be constructed for arbitrary dimension \( d \) of local quantum spaces:

\[ R(\lambda) = \frac{1}{2} \left[ \begin{array}{cc} |(1 + \omega)| & \frac{1}{2} \int \frac{d\omega}{1 + \omega} \ln(1 + \omega) d\omega \\
0 & \frac{1}{2} \int \frac{d\omega}{1 + \omega} \ln(1 + \omega) d\omega \end{array} \right] \]

The discrete functional equation

\[ \ln(\lambda_0) = \frac{1}{2} \int \frac{d\omega}{1 + \omega} \ln(1 + \omega) d\omega \]

The integral equations are obtained via the Temperley-Lieb equivalence for periodic boundary conditions from those of the XXZ chain with \( \lambda = \omega = 1 / 2 \) and a global twist angle \( \varphi = \pi \) [8].

We find a similar structure for the n-site density operator as in the XXX case:

\[ D_{\lambda_1, \lambda_2} = \frac{1}{2} \left( \int \frac{d\omega}{1 + \omega} \omega \right) \omega^{\lambda_1, \lambda_2} \]

with the operator \( f(2, \omega) \) st in the form

\[ D_{\lambda_1, \lambda_2} = \frac{1}{2} \left( \int \frac{d\omega}{1 + \omega} \omega \right) \omega^{\lambda_1, \lambda_2} \]

The fact that the density operator \( D_{\lambda} \) is an element of the Temperley-Lieb algebra is due to the special condition \( \omega = 0 \). Furthermore, if this condition does not hold two independent nearest neighbor correlators are needed to state the general solution (like for the XXZ model).

7 Summary and outlook

The discrete functional equation for finite Trotter number gives a unique characterization of \( D_{\lambda_1, \lambda_2} \).

The algebraic construction for \( T = 0 \) exploiting the structure coefficients of [3] can be directly used for finite temperature.

Application: computation of temperature dependent entanglement entropy, low-temperature asymptotics etc. [2]

For the case of higher spin, see the talk of A. Klümper on Friday.

References