On elliptic solutions to the reflection equation

RAQIS 2005, 9/9/2005

Yuji Yamada, Rikkyo University

$$R^{12}(u_1 - u_2)K^1(u_1)R^{21}(u_1 + u_2)K^2(u_2)$$
$$= K^2(u_2)R^{12}(u_1 + u_2)K^1(u_1)R^{21}(u_1 - u_2)$$

We consider the reflection equation associate to
Belavin's $\mathbf{Z}_N$-symmetric vertex model $R(u)$, which is a solution to YBE

$$R^{01}(u_0-u_1)R^{02}(u_0-u_2)R^{12}(u_1-u_2)$$
$$= R^{12}(u_1-u_2)R^{02}(u_0-u_2)R^{01}(u_0-u_1) \quad \in \mathrm{End}(\mathbf{C}^N\otimes\mathbf{C}^N\otimes\mathbf{C}^N).$$

$$R(u)=\frac{1}{N}\sum_{\alpha,\beta=0}^{N-1}\frac{\vartheta\begin{bmatrix}\alpha/N+1/2\\ \beta/N+1/2\end{bmatrix}\left(u+\frac{\eta}{N}\,\middle|\,\tau\right)}{\vartheta\begin{bmatrix}\alpha/N+1/2\\ \beta/N+1/2\end{bmatrix}\left(\frac{\eta}{N}\,\middle|\,\tau\right)}h^{-\alpha}g^{\beta}\otimes g^{-\beta}h^{\alpha},$$

$$ge_j=\omega^j e_j,\quad he_j=e_{j+1}\quad \left(\frac{\text{Ex. } N=3,\ \omega^3=1}{g=\begin{pmatrix}1&&\\&\omega&\\&&\omega^2\end{pmatrix},\quad h=\begin{pmatrix}&&1\\1&&\\&1&\end{pmatrix}}\right)$$

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(u\,|\,\tau)=\sum_{m\in\mathbf{Z}}\exp\left[\sqrt{-1}\pi\left((m+a)^2\tau+2(m+a)(u+b)\right)\right]$$
$$\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(u\,|\,\tau)=\theta_1(u) : \text{an odd function},$$

Study again why Belavin's $R(u)$ is a solution to YBE!

Belavin's $R(u)$ is characterized by the following conditions.

$\mathbf{e}[x] := \exp(\sqrt{-1}\pi x)$

1. $(g \otimes g)^{-1} R(u) (g \otimes g) = (h \otimes h)^{-1} R(u) (h \otimes h) = R(u)$,
2. $R(u+1) = \mathbf{e}[-1] (g \otimes 1)^{-1} R(u) (g \otimes 1)$,
3. $R(u+\tau) = \mathbf{e}\left[-\tau - 2\left(u + \frac{\eta}{N} + \frac{1}{2}\right)\right] (1 \otimes h)^{-1} R(u) (1 \otimes h)$,
4. $R(0) = P$, $P(x \otimes y) = y \otimes x$ for all $x, y \in \mathbf{C}^n$,
5. $R(u)$ is holomorphic in u.

$\Downarrow$

$$R(u) = \frac{1}{N} \sum_{\alpha,\beta=0}^{N-1} \frac{\vartheta\begin{bmatrix} \alpha/N + 1/2 \\ \beta/N + 1/2 \end{bmatrix}\left(u + \frac{\eta}{N} \,\middle|\, \tau\right)}{\vartheta\begin{bmatrix} \alpha/N + 1/2 \\ \beta/N + 1/2 \end{bmatrix}\left(\frac{\eta}{N} \,\middle|\, \tau\right)} h^{-\alpha} g^{\beta} \otimes g^{-\beta} h^{\alpha},$$

$\Downarrow$

This $R(u)$ satisfied the unitary relation

$\Downarrow$

This $R(u)$ satisfied the YBE!

When the unitary relation $PR(u)PR(-u) = \rho(u) \cdot Id$ holds, we immediately obtain the zeros of $(L.H.S) - (R.H.S)$ of the Yang-Baxter equation.
So the YBE is proved.

The unitary relation plays the key role!

We define $R_{(p,q)}(u)$ by

$$R_{(p,q)}(u) = \frac{1}{N} \sum_{\alpha,\beta=0}^{N-1} \frac{\vartheta \begin{bmatrix} p\alpha/N + 1/2 \\ q\beta/N + 1/2 \end{bmatrix} \left(u + \frac{\eta}{N} \,|\, \tau \right)}{\vartheta \begin{bmatrix} p\alpha/N + 1/2 \\ q\beta/N + 1/2 \end{bmatrix} \left(\frac{\eta}{N} \,|\, \tau \right)} h^{-\alpha} g^{\beta} \otimes g^{-\beta} h^{\alpha},$$

The original $R(u)$ is $R_{(1,1)}(u)$. Then we have

Lemma 1 *When p and q are coprime to N,*
$PR_{(p,q)}(u)PR_{(p,q)}(-u) = \rho(u)Id$ *is equivalent to* $PR(u)PR(-u) = \rho(u)Id$. □

Proposition 1 (Y.Quano 2000) *If* $K(u)$ *satisfies*

1. $K(u+1) = -g^{-1}K(u)g^{-1}$,

2. $K(u+\tau) = \mathbf{e}\left[-\tau - 2\left(u+\xi+\frac{1}{2}\right)\right] hK(u)h$,

3. $K(0) = Id$,

4. $K(u)$ *is holomorphic in* u,

and also the unitary relation

$$K(u)K(-u) = \rho_1(u)Id$$

Then $K(u)$ *is a solution to the reflection equation.* □

To construct such $K(u)$, we define the algebra isomorphism ϕ_0 between

$$\mathcal{A} := \langle h^{-a}g^b \otimes g^{-b}h^a \,|a,b \in \mathbf{Z}/N\mathbf{Z}\rangle \subset \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N)$$

and

$$\mathrm{End}(\mathbf{C}^N) = \langle h^a g^{-b} h^a \,|a,b \in \mathbf{Z}/N\mathbf{Z}\rangle.$$

by

$$\phi_0 : \mathcal{A} \longrightarrow \mathrm{End}(\mathbf{C}^N), \quad \phi_0(h^{-a}g^b \otimes g^{-b}h^a) = h^a g^{-b} h^a$$

The map $\phi_0 : \mathcal{A} \longrightarrow \mathrm{End}(\mathbf{C}^N)$ is an isomorphism just because $gh = \omega hg$.

$$R_{(2,2)}(u) = \frac{1}{N}\sum_{\alpha,\beta=0}^{N-1} \frac{\vartheta\begin{bmatrix} 2\alpha/N + 1/2 \\ 2\beta/N + 1/2 \end{bmatrix}\left(u + \frac{\eta}{N}\,|\,\tau\right)}{\vartheta\begin{bmatrix} 2\alpha/N + 1/2 \\ 2\beta/N + 1/2 \end{bmatrix}\left(\frac{\eta}{N}\,|\,\tau\right)} h^{-\alpha}g^{\beta} \otimes g^{-\beta}h^{\alpha} \in \mathcal{A} \text{ satisfies}$$

(i) $R_{(2,2)}(u+1) = \mathbf{e}[-1](g^{-1} \otimes g)R_{(2,2)}(u)(g \otimes g^{-1})$,

(ii) $R_{(2,2)}(u+\tau) = \mathbf{e}\left[-\tau - 2\left(u + \frac{\eta}{N} + \frac{1}{2}\right)\right](h \otimes h{-}1)R_{(2,2)}(u)(h^{-1} \otimes h)$,

(iii) $R_{(2,2)}(0) = P$, $P(x \otimes y) = y \otimes x$ for all $x, y \in \mathbf{C}^n$,

(iv) $R_{(2,2)}(u)$ is holomorphic in u.

(v) the unitary relation $PR_{(2,2)}(u)PR_{(2,2)}(-u) = \rho_1 Id$.

We consider the image of $P \cdot R_{(2,2)}(u)$ under ϕ_0, $K(u) := \phi_0\left(P \cdot R_{(2,2)}(u)\right)$.

(i) $K(u+1) = -g^{-1}K(u)g^{-1}$,

(ii) $K(u+\tau) = \mathbf{e}\left[-\tau - 2\left(u + \frac{\eta}{N} + \frac{1}{2}\right)\right] hK(u)h$

(iii) $K(0) = \mathcal{K}$, $\mathcal{K}e_j = e_{-j}$ where $\langle e_j \rangle$ is a O.N.B of $\mathbf{C}^N$,

(iv) $K(u)$ is holomorphic in u.

(v) the unitary relation $K(u)K(-u) = \rho_1(u)Id$.

Theorem 1 *When N is odd, $K(u) := \phi_0 \left(P \cdot R_{(2,2)}(u) \right)$ is a solution to the reflection equation*

$$\begin{aligned} & R^{12}_{(1,1)}(u_1 - u_2) K^1(u_1) R^{21}_{(1,1)}(u_1 + u_2) K^2(u_2) \\ = \ & K^2(u_2) R^{12}_{(1,1)}(u_1 + u_2) K^1(u_1) R^{21}_{(1,1)}(u_1 - u_2) \end{aligned}$$

where the isomorphism

$$\phi_0 : \mathcal{A} \longrightarrow \mathrm{End}(\mathbf{C}^N)$$

is defined by

$$\phi_0(h^{-a} g^b \otimes g^{-b} h^a) = h^a g^{-b} h^a$$

∎

where $he_j = e_{j+1}$, $ge_j = w^j e_j$, $gh = \omega hg$ and

$$\begin{aligned} \mathcal{A} \ &:= \ \langle h^{-a} g^b \otimes g^{-b} h^a \, | a, b \in \mathbf{Z}/N\mathbf{Z} \rangle \subset \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N) \\ \mathrm{End}(\mathbf{C}^N) \ &= \ \langle h^a g^{-b} h^a \, | a, b \in \mathbf{Z}/N\mathbf{Z} \rangle . \end{aligned}$$

Proposition 2 *If* $K(u)$ *satisfies*

1. $K(u+1) = g^{-1}K(u)g^{-1}$,

2. $K(u+\tau) = \mathbf{e}[-4\tau - 2(4u+2\xi)]\, hK(u)h$,

3. $K(0) = Id$,

4. $K\left(\frac{1}{2}\right) = C_1 g^{-1},\ K\left(\frac{1}{2}\tau\right) = C_2 h,\ K\left(\frac{1}{2}+\frac{1}{2}\tau\right) = C_3 h g^{-1}$,

5. $K(u)$ *is holomorphic in* u,

and also the unitary relation

$$K(u)K(-u) = \rho_1(u)Id$$

Then $K(u)$ *is a solution to the reflection equation.* □

Where do new initial conditions $K\left(\frac{1}{2}\right)=C_1g^{-1}, K\left(\frac{1}{2}\tau\right)=C_2h, K\left(\frac{1}{2}+\frac{1}{2}\tau\right)=C_3hg^{-1}$ work? For example, the reflection eqation at $u_1=\frac{1}{2}$,

$$R^{12}(u_1-u_2)K^1(u_1)R^{21}(u_1+u_2)K^2(u_2)-K^2(u_2)R^{12}(u_1+u_2)K^1(u_1)R^{21}(u_1-u_2)$$

$$= R^{12}(1/2-u_2)K^1(1/2)R^{21}(1/2+u_2)K^2(u_2)-K^2(u_2)R^{12}(1/2+u_2)K^1(1/2)R^{21}(1/2-u_2)$$

$$= R^{12}(1-1/2-u_2)K^1(1/2)R^{21}(1/2+u_2)K^2(u_2)$$
$$-K^2(u_2)R^{12}(1/2+u_2)K^1(1/2)R^{21}(1-1/2-u_2)$$

$$= C_1(g^{-1}\otimes 1)R^{12}(-1/2-u_2)(g\otimes 1)(g^{-1}\otimes 1)R^{21}(1/2+u_2)K^2(u_2)$$
$$-C_1K^2(u_2)R^{12}(1/2+u_2)(g^{-1}\otimes 1)(1\otimes g^{-1})R^{21}(-1/2-u_2)(1\otimes g)$$

$$= C_1(g^{-1}\otimes 1)R^{12}(-1/2-u_2)R^{21}(1/2+u_2)K^2(u_2)$$
$$-C_1K^2(u_2)R^{12}(1/2+u_2)R^{21}(-1/2-u_2)(g^{-1}\otimes 1)$$
$$= C_1\rho_1(1/2+u_2)(g^{-1}\otimes 1)K^2(u_2)-C_1\rho_1(1/2+u_2)K^2(u_2)(g^{-1}\otimes 1)=0$$

These conditions give enough number of zeros to vanish $(L.H.S.)-(R.H.S.)$ of the reflection equation.

To construct $K(u)$ which satisfies

1. $K(u+1) = g^{-1}K(u)g^{-1}$,
2. $K(u+\tau) = \mathrm{e}[-4\tau - 2(4u + 4\xi)]\, hK(u)h$,
3. $K(0) = Id$,
4. $K\left(\frac{1}{2}\right) = C_1 g^{-1}$, $K\left(\frac{1}{2}\tau\right) = C_2 h$, $K\left(\frac{1}{2} + \frac{1}{2}\tau\right) = C_3 h g^{-1}$,
5. $K(u)$ is holomorphic in u,
6. the unitary relation $K(u)K(-u) = \rho_2(u)Id$.

we consider $R_{(1,n+1)}(2u,\xi)$. Because $n+1$ is coprime to $N = 2n+1$, $R_{(1,n+1)}(2u,\xi)$ satisfies the unitary relation.

$$R_{(1,n+1)}(2u,\xi) = \frac{1}{N}\sum_{\alpha,\beta=0}^{N-1} \frac{\vartheta\begin{bmatrix} \alpha/N + 1/2 \\ (n+1)\beta/N + 1/2 \end{bmatrix}\left(2u + \frac{\xi}{N} \,|\, \tau\right)}{\vartheta\begin{bmatrix} \alpha/N + 1/2 \\ (n+1)\beta/N + 1/2 \end{bmatrix}\left(\frac{\xi}{N} \,|\, \tau\right)} h^{-\alpha} g^{\beta} \otimes g^{-\beta} h^{\alpha}$$

$$R_{(1,n+1)}(2u,\xi) = \frac{1}{N}\sum_{\alpha,\beta=0}^{N-1} \frac{\vartheta\left[\begin{array}{c}\alpha/N+1/2\\(n+1)\beta/N+1/2\end{array}\right]\left(2u+\frac{\xi}{N}\,|\,\tau\right)}{\vartheta\left[\begin{array}{c}\alpha/N+1/2\\(n+1)\beta/N+1/2\end{array}\right]\left(\frac{\xi}{N}\,|\,\tau\right)} h^{-\alpha}g^{\beta}\otimes g^{-\beta}h^{\alpha}$$

Let us denote $(N = 2n+1)$

$$\begin{aligned}
K_1(u) &:= \phi_0\left(PR_{(1,n+1)}(2u,\xi)\right),\\
K_2(u) &:= \phi_0\left(PR_{(1,n+1)}\left(2u,\xi-\frac{N}{2}\right)\right),\\
K_3(u) &:= \phi_0\left(PR_{(1,n+1)}\left(2u,\xi-\frac{N}{2}-\frac{N}{2}\tau\right)\right),\\
K_4(u) &:= \phi_0\left(PR_{(1,n+1)}\left(2u,\xi-\frac{N}{2}\tau\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\phi_0 &: \mathcal{A}\longrightarrow \mathrm{End}(\mathbf{C}^N), \quad \phi_0\left(h^{-a}g^{b}\otimes g^{-b}h^{a}\right) = h^{a}g^{-b}h^{a}\\
\mathcal{A} &:= \langle h^{-a}g^{b}\otimes g^{-b}h^{a}\,|a,b\in\mathbf{Z}/N\mathbf{Z}\rangle \subset \mathrm{End}(\mathbf{C}^N\otimes\mathbf{C}^N)\\
\mathrm{End}(\mathbf{C}^N) &= \langle h^{a}g^{-b}h^{a}\,|a,b\in\mathbf{Z}/N\mathbf{Z}\rangle.
\end{aligned}$$

Then $K_i(u)$ $(i = 1, 2, 3, 4)$ satisfy

1. $K_i(u+1) = g^{-1} K_i(u) g^{-1}$,

2. $K_i(u+\tau) = \mathrm{e}\,[\,-4\tau - 2(4u + 4\xi)\,]\, h K_i(u) h$,

3. $K(0) = Id$,

4. $K_i\left(\frac{1}{2}\right) = C_1^{(i)} g^{-1},\ K_i\left(\frac{1}{2}\tau\right) = C_2^{(i)} h,\ K_i\left(\frac{1}{2} + \frac{1}{2}\tau\right) = C_3^{(i)} h g^{-1}$,
$\det |C_l^{(i)}| \neq 0$ $(C_0^{(i)} = 1)$

5. $K_i(u)$ is holomorphic in u,

Furthermore, because $K_i(u) K_j(-u) = K_j(u) K_i(-u)$ holds for $i \neq j$, $K(u) = \sum_{i=1}^{4} \alpha_i K_i(u)$ $(\sum_{i=1}^{4} \alpha_i = 1$ for $K(0) = Id)$ satisfies the unitary relation

$$K(u) K(-u) = \rho_3(u) Id$$

Dimension of the solution space

Conversely, if $K(u)$ satisfies $K(u+1) = g^{-1}K(u)g^{-1}$

$$K(u+\tau) = \mathbf{e}\left[-4\tau - 2\left(4u + \frac{2\xi}{N}\right)\right] hK(u)h$$

The function $y_{00}(u)$ in the expression

$$K(u) = \sum_{\alpha,\beta=0}^{N-1} y_{\alpha\beta}(u) h^{\alpha} g^{-2\beta} h^{\alpha},$$

has $4N^2$ zeros in $\mathbf{C}/(N\mathbf{Z} \oplus N\mathbf{Z}\tau)$. And the conditions

$$K(0) = Id, \quad K\left(\frac{1}{2}\right) = C_1 g^{-1}, \quad K\left(\frac{1}{2}\tau\right) = C_2 h, \quad K\left(\frac{1}{2} + \frac{1}{2}\tau\right) = C_3 h g^{-1},$$

and the quasiperiodicity let us find $4N^2 - 4$ zeros of $y_{00}(u)$

$$y_{00}\left(\frac{k}{2} + \frac{l}{2}\tau\right) = 0 \text{ for } k, i, \in \mathbf{Z}/(2N\mathbf{Z}) \text{ except } (k.l) = (0,0), (1,0), (0,1), (1,1)$$

So, the dimension of linear span of $y_{00}(u)$ is 4.

Theorem 2 *When N is odd, the matrices characterized by*

1. $K(u+1) = g^{-1}K(u)g^{-1}$,
2. $K(u+\tau) = \mathrm{e}[-4\tau - 2(4u+4\xi)]\,hK(u)h$,
3. $K(0) = Id$,
4. $K\left(\frac{1}{2}\right) = C_1 g^{-1},\ K\left(\frac{1}{2}\tau\right) = C_2 h,\ K\left(\frac{1}{2}+\frac{1}{2}\tau\right) = C_3 h g^{-1}$,
5. $K(u)$ *is holomorphic in* u,

has 4 parameters, and are expressed in terms of Belavin's R-matrix $R_{(1,n+1)}(2u,\xi)$

$$K(u) = \phi_0 \begin{pmatrix} \alpha_1 R_{(1,n+1)}(2u,\xi) & +\alpha_2 R_{(1,n+1)}\left(2u, \xi - \frac{N}{2}\right) \\ +\alpha_3 R_{(1,n+1)}\left(2u, \xi - \frac{N\tau}{2}\right) & +\alpha_4 R_{(1,n+1)}\left(2u, \xi - \frac{N}{2} - \frac{N\tau}{2}\right) \end{pmatrix}$$

It satisfies the unitary relation, and is a solution to the reflection equation.

$$\begin{aligned} & R^{12}_{(1,1)}(u_1-u_2)K^1(u_1)R^{21}_{(1,1)}(u_1+u_2)K^2(u_2) \\ = \ & K^2(u_2)R^{12}_{(1,1)}(u_1+u_2)K^1(u_1)R^{21}_{(1,1)}(u_1-u_2) \end{aligned}$$

Theorem 1 *When N is odd, the matrices characterized by*

1. $K(u+1) = -g^{-1}K(u)g^{-1}$,
2. $K(u+\tau) = \mathbf{e}\left[-\tau - 2(u+\xi+\frac{1}{2})\right] hK(u)h$,
3. $K(0) = Id$,
4. $K(u)$ *is holomorphic in* u,

has 1 parameter, and are expressed in terms of Belavin's R-matrix $R_{(2,2)}(u,\xi)$

$$K(u) = \phi_0\left(R_{(2,2)}(u,\xi)\right)$$

It satisfies the unitary relation, and is a solution to the reflection equation.

$$\begin{aligned} & R^{12}_{(1,1)}(u_1-u_2)K^1(u_1)R^{21}_{(1,1)}(u_1+u_2)K^2(u_2) \\ = \; & K^2(u_2)R^{12}_{(1,1)}(u_1+u_2)K^1(u_1)R^{21}_{(1,1)}(u_1-u_2) \end{aligned}$$

$$R_{(p,q)}(u) = \frac{1}{N}\sum_{\alpha,\beta=0}^{N-1} \frac{\vartheta\begin{bmatrix} p\alpha/N + 1/2 \\ q\beta/N + 1/2 \end{bmatrix}\left(u + \frac{\eta}{N} \,|\, \tau\right)}{\vartheta\begin{bmatrix} p\alpha/N + 1/2 \\ q\beta/N + 1/2 \end{bmatrix}\left(\frac{\eta}{N} \,|\, \tau\right)} h^{-\alpha} g^{\beta} \otimes g^{-\beta} h^{\alpha},$$

The original $R(u)$ is $R_{(1,1)}(u)$.

$$\begin{aligned} \phi_0 \quad &: \quad \mathcal{A} \longrightarrow \mathrm{End}(\mathbf{C}^N), \quad \phi_0\left(h^{-a} g^{b} \otimes g^{-b} h^{a}\right) = h^{a} g^{-b} h^{a} \\ \mathcal{A} \quad &:= \quad \langle h^{-a} g^{b} \otimes g^{-b} h^{a} \,|\, a, b \in \mathbf{Z}/N\mathbf{Z}\rangle \subset \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N) \\ \mathrm{End}(\mathbf{C}^N) \quad &= \quad \langle h^{a} g^{-b} h^{a} \,|\, a, b \in \mathbf{Z}/N\mathbf{Z}\rangle. \end{aligned}$$