RAQIS 2005

“QUANTUM” DEFORMATIONS OF CLASSICAL DYNAMICAL SYSTEMS

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PLB 610 (2005), 107-114; JPA (to appear)

- AIM 1: to show the intimate relation between algebraic notions and quantities (namely $q-$Poisson coalgebras) and geometric ones(integrable geodesic motions on 2D manifolds with constant and non-constant curvature)
- TOOLS: Hopf-algebra structure of $Non - Standard$ $q-$deformations
- AIM 2: to cast Classical (and hopefully Quantum) Hamiltonian Systems integrable by “coalgebra symmetry” approach in the more familiar Lax formalism
- TOOLS: usual tricks of soliton theory and a bit of guesswork: much has still to be done...

PLAN OF THE LECTURE

1. Hamiltonians with co-algebra symmetry
2. Non-Standard deformations
3. Integrable Hamiltonians and non-constant curvature
4. Super-integrable Hamiltonians and constant curvature
5. More degrees of freedom. Evolution Equations and their solution
6. Reminder of Classical $\mathcal{U}_q(sl(2))$ Gaudin Hamiltonian and cluster variables
7. Lax pair for Classical $\mathcal{U}_q(sl(2))$ Gaudin model

I. Poisson Coalgebra $(sl(2,\mathbb{C}), \Delta)$

$$sl(2,\mathbb{C}) := \{J_3, J_+, J_-\}$$

$$\{J_3, J_\pm\} = \pm 2J_\pm$$

$$\{J_+, J_-\} = 4J_3$$

- Δ : co-associative Poisson Homomorphism:

$$\Delta : (sl(2,\mathbb{C}) \longrightarrow (sl(2,\mathbb{C}) \oplus (sl(2,\mathbb{C})$$

$$\Delta(J_k) = J_k \oplus I + I \oplus J_k$$

- One particle symplectic realization:

$$J_-^{(1)} = q_1^2 \quad J_+^{(1)} = p_1^2 + b_1/q_1^2 \quad J_3^{(1)} = q_1 p_1$$

- Casimir function

$$\mathcal{C}^{(1)} = J_- J_+ + J_3^2 = b_1$$

- From 1- to 2- (and to many-) particle symplectic realization through Δ

$$J_-^{(2)} = q_1^2 + q_2^2 \quad J_+^{(2)} = p_1^2 + p_2^2 + b_1/q_1^2 + b_2/q_2^2$$

$$J_3^{(2)} = q_1 p_1 + q_2 p_2$$

- Fundamental property:

  Any smooth function $\mathcal{H}^{(2)} = \mathcal{H}(J_-^{(2)}, J_+^{(2)}, J_3^{(2)}$ (*) defines a completely integrable two-particle system, as it is equipped with the extra-integral of motion $\mathcal{C}^{(2)}$,reading:

$$\mathcal{C}^{(2)} = \Delta(\mathcal{C}) =$$

$$(q_1 p_2 - q_2 p_1)^2 + (\frac{b_1}{q_1^2} + \frac{b_2}{q_2^2})(q_1^2 + q_2^2)$$

- Hence, integrability of <u>any</u> Hamiltonian (*) is merely a consequence of coalgebra symmetry

It is worth to notice that, moreover, there are exceptional hamiltonians of type (*) which are Superintegrable (SI), namely, a further integral of motion exists:

$$\{\mathcal{H}^{(2)}, \mathcal{I}^{(2)}\} = 0$$

Example: if we consider a generic hamiltonian of the form:

$$\mathcal{H} = \frac{1}{2} J_+ \mathcal{F}(J_-)$$

for linear $\mathcal{F}$ we get a super-integrable system.

II. Integrable Systems through Non-Standard Deformations of $(sl(2,\mathbb{C}),\Delta)$

- Deformed PB:

$$\{J_3,J_+\}=2J_+\cosh zJ_- \qquad \{J_3,J_-\}=-2\,\frac{\sinh zJ_-}{z} \qquad \{J_-,J_+\}=4J_3$$

- Casimir function

$$\mathcal{C}_z=\frac{\sinh zJ_-}{z}J_+-J_3^2$$

- Deformed Coproduct

$$\Delta_z(J_-)=J_-\otimes 1+1\otimes J_- \qquad \Delta_z(J_i)=J_i\otimes \mathrm{e}^{zJ_-}+\mathrm{e}^{-zJ_-}\otimes J_i \quad i=+,3$$

z: real deformation parameter

- One and two particle symplectic realization

One-particle:

$$J_- = q_1^2 \qquad J_3 = \frac{\sinh z q_1^2}{z q_1^2} q_1 p_1$$
$$J_+ = \frac{\sinh z q_1^2}{z q_1^2} p_1^2$$

Two-particle:

$$J_- = q_1^2 + q_2^2 \qquad J_3 = \frac{\sinh z q_1^2}{z q_1^2}\, q_1 p_1\, \mathrm{e}^{z q_2^2} + \frac{\sinh z q_2^2}{z q_2^2}\, q_2 p_2\, \mathrm{e}^{-z q_1^2}$$
$$J_+ = \left(\frac{\sinh z q_1^2}{z q_1^2}\, p_1^2 + \frac{z b_1}{\sinh z q_1^2} \right) \mathrm{e}^{z q_2^2} + \left(\frac{\sinh z q_2^2}{z q_2^2}\, p_2^2 + \frac{z b_2}{\sinh z q_2^2} \right) \mathrm{e}^{-z q_1^2}$$

Two-particle Casimir:

$$\mathcal{C}_z = \frac{\sinh z q_1^2}{z q_1^2} \frac{\sinh z q_2^2}{z q_2^2} (q_1 p_2 - q_2 p_1)^2\, \mathrm{e}^{-z q_1^2} \mathrm{e}^{z q_2^2} + (b_1 \mathrm{e}^{2 z q_2^2} + b_2 \mathrm{e}^{-2 z q_1^2})$$
$$+ \left(b_1 \frac{\sinh z q_2^2}{\sinh z q_1^2} + b_2 \frac{\sinh z q_1^2}{\sinh z q_2^2} \right) \mathrm{e}^{-z q_1^2} \mathrm{e}^{z q_2^2} .$$

Most general integrable deformation of the free motion in $\mathbb{E}^2$ ($\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2)$) :

$$\mathcal{H} = \frac{1}{2} J_+ f(zJ_-)$$

Simplest choice: $f(x) = 1$: however, not superintegrable!.

Superintegrable hamiltonian:

$$\mathcal{H}_z^S = \frac{1}{2} J_+ \exp(zJ_-)$$

i.e : $f(x) = x$

Extra-integral:

$$\mathcal{I}_z = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 \exp(q_1^2) = J_+^{(1)} \exp(zJ_-^{(1)})$$

$\mathcal{H}_z^S, \mathcal{I}_z, \mathcal{C}_z$: functionally independent

Natural interpretation:

Hamiltonians of the form $J_+ f(zJ_-)$ are deformed kinetic energies:

$$\mathcal{H}_z = \mathcal{T}_z(q_i, p_i)$$

We will show:

1. $\mathcal{H}_z^I$: geodesic motion in 2D Riemannian space or 1+1 rel. space-time, with curvature depending both on z and on the point (q,p);
2. $\mathcal{H}_z^S$: geodesic motion … with curvature depending just on z

III. Integrable Deformations and Non-Constant Curvature

Let $\mathcal{H}_z^I(q_i, p_i) \quad \rightarrow \mathcal{T}_z^I(q_i, \dot{q}_i) \quad$ (Legendre Transformation):

$$\mathcal{T}_z^I(q_i, \dot{q}_i) = \frac{1}{2}\left(\frac{(\dot{q}_1)^2 \exp(-z(\dot{q}_2)^2}{s_z(q_1^2)} + \frac{(\dot{q}_2)^2 \exp(z(\dot{q}_1)^2}{s_z(q_2^2)}\right)$$

$$s_z(x) := \frac{\sin(zx)}{zx}$$

yields a geodesic flow on a 2D space.

- Metric:

$$ds^2 \equiv \frac{\exp(-z(\dot{q}_2)^2}{s_z(q_1^2)} dq_1^2 + \frac{\exp(z(\dot{q}_1)^2}{s_z(q_2^2)} dq_2^2 :=$$
$$g_{11}(q) dq_1^2 + g_{22}(q) dq_2^2$$

- Gaussian curvature:

$$K = -\frac{1}{(g_{11}g_{22})^{\frac{1}{2}}}\{\frac{\partial}{\partial q_1}(g_{11}^{-\frac{1}{2}}\frac{\partial}{\partial q_1}g_{22}^{\frac{1}{2}}) + \frac{\partial}{\partial q_2}(g_{22}^{-\frac{1}{2}}\frac{\partial}{\partial q_2}g_{11}^{\frac{1}{2}})\} =$$

$$-z\sinh[z(q_1^2+q_2^2)]$$

K: negative and nonconstant!

Notice: To give a nonconstant curvature, the exponentials appearing in the deformed coproducts are essential!

Geometry is better seen through a change of variables.

$$\cosh(\lambda_1\rho) = \exp z(q_1^2 + q_2^2) \quad (\rho > 0)$$

$$\sin^2(\lambda_2\theta) = \frac{\exp(2zq_1^2) - 1}{\exp z(q_1^2 + q_2^2) - 1}$$

Remarks

- We have set $z = \lambda_1^2$ and we have introduced a new real parameter λ_2, related with the *signature* of the metric.
- The new variable $\cosh(\lambda_1\rho)$ is a collective variable, function of $\Delta(J_-)$; its role will be further specified later).
- The zero-deformation limit (improperly called the "classical limit") $z \to 0$ is in fact the flat limit $K \to 0$. In this limit $\rho \to 2(q_1^2 + q_2^2)$, $\sin^2(\lambda_2\theta) \to \frac{q_1^2}{q_1^2+q_2^2}$

Metric in the new variables:

$$ds^2 = \frac{1}{\cosh(\rho)}(d\rho^2 + \frac{\lambda_2^2}{\lambda_1^2}\sinh^2(\lambda_1\rho)d\theta^2) = \frac{1}{\cosh(\rho)}ds_0^2$$

$$ds_0^2 : \mathrm{so-called} \quad \mathrm{CK\ (Cayley-Klein)\ metric.}$$

$$K = K(\rho) = -\frac{1}{2}\lambda_1^2\frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)}$$

$$z \in \mathbb{R}^+ : K < 0; \qquad z \in \mathbb{R}^- : K\mathrm{periodic}$$

Kinetic energy and Hamiltonian:

$$\mathcal{T}_z^I(q,\dot{q}) = \frac{1}{2}\frac{1}{\cosh(\lambda_1\rho)}((\dot{\rho})^2 + \frac{\lambda_2^2}{\lambda_1^2}\sinh^2(\lambda_1\rho)(\dot{\theta})^2))$$

$$\mathcal{H}_z^I(q,p) = \frac{1}{2}\cosh(\lambda_1\rho)(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2}\sinh^{-2}(\lambda_1\rho)(p_\theta)^2)$$

Moreover, as $(p_\theta)^2 = \mathcal{C}_z^I$, the usual reduction to the radial coordinate can be performed.

Specializations:

- $\lambda_2 \in \mathbb{R}$: $z \in \mathbb{R}^+ : \ \mathrm{def.\ Hyperbolic - space}; \quad z \in \mathbb{R}^- : \ \mathrm{def.\ sphere}$
- $\lambda_2 \in i\mathbb{R}$: $z \in \mathbb{R}^+ : \ \mathrm{def.\ DS - space}; \quad z \in \mathbb{R}^- : \ \mathrm{def.\ ADS - space}$

IV. Super-Integrable Deformations and Constant Curvature

- We start from the Superintegrable Hamiltonian:

$$\mathcal{H}_z^S = \frac{1}{2} J_+ \exp(zJ_-)$$

- Legendre Transform → the two-body "free" Lagrangian (Kinetic energy):

$$\mathcal{T}_z^S(q,\dot{q}) = \frac{1}{2}\Big(\frac{\exp(-z(q_1^2+2q_2^2))}{s_z(q_1^2)}(\dot{q}_1)^2 + \frac{\exp(-zq_2^2)}{s_z(q_2^2)}(\dot{q}_2)^2\Big)$$

- Associated metric:

$$ds^2 = \Big(\frac{\exp(-z(q_1^2+2q_2^2))}{s_z(q_1^2)}(\dot{q}_1)^2 + \frac{\exp(-zq_2^2)}{s_z(q_2^2)}(\dot{q}_2)^2\Big)$$

- Gaussian curvature:

$$K(q,z) = z \quad = \quad \text{const.}$$

- Change of variables (as before):

$$ds^2 = \frac{1}{\cosh^2(\lambda_1\rho)}(d\rho^2 + \frac{\lambda_2^2}{\lambda_1^2} sinh^2(\lambda_1\rho)d\theta^2) =$$

$$= \frac{1}{\cosh^2(\lambda_1\rho)} ds_0^2$$

- New radial variable:

$$r = \int_0^\rho \frac{dx}{\cosh(\lambda_1 x)}$$

whence: $\tan(\lambda_1 r) = \sinh(\lambda_1\rho)$; $\quad \cos(\lambda_1 r) = \frac{1}{\cosh(\lambda_1\rho)}$

Finally:

$$\mathcal{T}_z^S = \frac{1}{2}(\dot{r})^2 + \frac{\lambda_2^2}{\lambda_1^2}\sin^2(\lambda_1 r)(\dot{\theta})^2)$$
$$\mathcal{H}_z^S = \frac{1}{2}(p_r)^2 + \frac{\lambda_1^2}{\lambda_2^2\sin^2(\lambda_1 r)}(p_\theta)^2)$$

Integrals of motion:

$$\mathcal{C}_z^S = p_\theta^2; \quad \mathcal{I}_z^S = (sin(\lambda_2\theta)p_r + \frac{\lambda_1}{\lambda_2}\frac{\cos(\lambda_2\theta)}{\tan(\lambda_1 r)}p_\theta^2)^2$$

Comment: the change of variable $\rho \to r$ through $dr = \frac{d\rho}{(\cosh(\lambda_1\rho))^{\frac{1}{2}}}$ is of course admissible even in the non-superintegrable case; however, with negligible advantage.

V. Many-Body Case; preliminary results

Co-algebra symmetry → N-body integrable version.

Example: N-body version of the simplest Hamiltonian:

$$\mathcal{H}_z^{I(N)} = \frac{1}{2}\sum_{j=1}^{N} s_z(q_j^2)p_j^2 \exp(z\sum_{k\neq j}\mathrm{sgn}(k-j)q_k^2)$$

Again we get a "free" Lagrangian:

$$\mathcal{T}_z^{I(N)} = \frac{1}{2}\sum_{i=1}^{N} \frac{(\dot{q}_i)^2 \exp(-z\sum_{k\neq j}\mathrm{sgn}(j-k)q_k^2)}{s_z(q_i^2)}$$

with the obvious corresponding metric.

Comment: Geometric interpretation more natural in terms of Hyperspherical (or analogous) coordinates. Work in progress (Ballesteros, Herranz

VI. Evolution Equations

Main advantage (and limitation) of dynamical systems with co-algebra symmetry:

For any N the N-body dynamics reduces to a two-body dynamics !

Indeed:

1. Take as $\mathcal{H}$ any fn $\mathcal{F}$ of the generators $J^{\pm}$, J_3 (stick to $sl(2)$), then:

$$\mathcal{H}^{(N)} = \Delta^{(N)}(\mathcal{F}) = \mathcal{F}(\Delta^{(N)}(J^{\pm}), \Delta^{(N)}(J_3))$$

2. Solve the (1-body !) eqs. of motion for the collective variables $\Delta^{(N)}(J^{\pm}), \Delta^{(N)}(J_3)$
3. Split your system in two *clusters*, corresponding to the coproducts, say, $\Delta^{(m)}$, $\Delta^{(N-m)}$: the dynamics is the functionally the same $\forall\ m$!. Only the coefficents may be different, arising from partial Casimirs $\delta^{(m)} := \Delta^{(m)}(C)$ which are by construction integrals of motion!

4. The N-body dynamics is then reduced to a two-body one. However, you can do more! For instance, you can eliminate dynamical variables arising from $\Delta^{(N-m)}$ in favor of $\Delta^{(m)}$ and $\Delta^{(N)}$, with suitable integration constants

5. Finally, you end up with a typical mean field dynamics: each cluster moves as a single particle in a field generated self-consistently by the individual constituents.

- Examples

1. E_1: Geodesic motion on constant curvature surfaces
2. E_2: Deformed Harmonic motion on constant curvature surfaces.
3. E_3: Geodesic motion on nonconstant curvature surfaces

EXAMPLE I

Let:

$$\mathcal{H} = J_+ \exp(zJ_-) = \frac{\exp(2zq^2 - 1)}{2zq^2} p^2$$

Define

$$a_i = J_{3,i} \exp(zJ_{-,i}); \quad b_i = J_{+,i} \exp(zJ_{-,i}); \quad c_i = J_{-,i}$$

$$\mathcal{C}_{z,i} = \exp(-2zc_i)(a_i^2 + b_i \frac{\exp(2zc_i) - 1}{2z})$$

We don't work with single particle variables, but first use:

$$
\begin{aligned}
&a = \Delta^{(2)}(a_1) = \Delta^{(2)}(J_{3,1})\exp(z\Delta^{(2)}J_{-,1}) = \\
&a_1 + \exp(2zc_1)a_2 \\
&b = \Delta^{(2)}(b_1) = b_1 + \exp(2zc_1)b_2 \ := \ \mathcal{H}_2 \\
&c = \Delta^{(2)}(c_1) = c_1 + c_2
\end{aligned}
$$

Then, turn to a_1, b_1, c_1

Remark: Geometric variables: $\cosh(\lambda_1\rho) = \exp(2zc) \quad \sin^2(\lambda_2\theta) = \frac{\exp(2zc_1)-1}{\exp(2zc)-1}$

According with the previous outlined strategy, we start by solving the simplest equation, involving collecting variables, then solve for single-particle dynamics

Evolution equations for collective variables:

$$
\begin{aligned}
&\dot{a} = 2b + 4a^2 = E + 4a^2 \\
&\dot{b} = 0 \\
&\dot{c} = 4a
\end{aligned}
$$

There are two cases, according to the sign of zE.

1. $2zE = \ \gamma^2 \ > 0, \quad \gamma \in \mathbb{R}$; then:

   $$\begin{aligned} & a = \frac{E}{\gamma} \tanh(2\gamma(t - t_0)) \\ & \cosh(\lambda_1 \rho) = \exp(2zc) = \cosh(2\gamma(t - t_0)) \end{aligned}$$

   Notice: The radius ρ grows linearly in time.

2. $2zE = \ -\gamma^2 \ < 0, \quad \gamma \in \mathbb{R}$. Hyperbolic functions are replaced by trigonometric ones. However, having to do with free motion, the energy E has to be taken as positive. So it is z that changes sign, and consequently again the radius ρ grows linearly in time.

The one-body variables obey the system of nonlinear equations:

$$\begin{aligned}
\dot{a}_1 &= 2b_1 + 4za_1^2 + 4b_2 \exp(2c_1 z)\frac{\exp(2c_1 z) - 1}{2z} \\
\dot{b}_1 &= 8za_1 \exp(2zc_1) b_2 \\
\dot{c}_1 &= 4a_1
\end{aligned}$$

which can be explicitly solved in terms of trigonometric/hyperbolic functions.

You may proceed as follows:

- From the second and the third equation, you get:

$$\exp(2zc_1) = \frac{E - b_1}{b_2} \quad b_2 : \text{constant of the motion}$$

- Then, you use the one-body Casimir $\delta^{(1)}$, such that:

$$exp(2zc_1) = \frac{za_1^2 + b_1}{z\delta^{(1)} + b_1}$$

  to eliminate a_1 in favor of b_1, c_1, finally getting the evolution equation for $\gamma_1 := \exp(2zc_1)$:

$$\dot{\gamma}_1 = 8\gamma_1\sqrt{zb_2(\gamma_1 - \gamma_+)(\gamma_1 - \gamma_-)}$$

  the parameters $\gamma_\pm$ being given in terms of the constants $b_2, \delta^{(1)}, E$.

- For $zE < 0, \gamma_\pm \in \mathbb{R}$ the solution is given in terms of trigonometric functions and reads:

$$\gamma_1 = \frac{\gamma_+\gamma_-}{\gamma_+ \cos^2(\sqrt{zE}(t - t_0) + \gamma_- \sin^2(\sqrt{zE}(t - t_0)}$$

EXAMPLE II

Let

$$\mathcal{H} = \exp(zJ_-)(J_+\omega^2\frac{\sinh(zJ_-)}{z})$$

It describes the motion of a particle in the field given by a $q-$deformed harmonic oscillator, on a surface with constant curvature.

The dynamical variables and the Casimir are defined as before. The equations for the collective variables are easily written down in terms of variables $a, b, \gamma := \exp(2zc) = \cosh(\lambda_1\rho)$.

$$\dot{a} = 2b + 4za^2 + \omega^2 \exp(2zc)(\frac{\exp(2zc) - 1}{z})$$
$$\dot{b} = -4\omega^2 a\gamma$$
$$\dot{\gamma} = 8za\gamma$$

Thanks to the integrals of motion $\mathcal{H}, \mathcal{C}_z$ one finally gets a first order evolution equation for γ ($\longrightarrow$ for ρ):

$$\dot{\gamma} = 8z\gamma\sqrt{\omega^2(\gamma - \gamma_+)(\gamma_- - \gamma)}$$

For suitable values of $\mathcal{H}, \mathcal{C}_z$ the motion for γ is periodic, expressed in terms of trigonometric functions, just as that for the one-body variables derived in the previous example, and confined in the interval $[\gamma_-, \gamma_+]$.

Following again the same path, one now considers evolution equations for single-particle variables:

$$\begin{aligned}
\dot{a}_1 &= 2b_1 + 4za_1^2 + 2\exp(2zc)\frac{1-2\exp(2c_1 z)}{2z}(2zb_2\exp(-2zc_2+\omega^2) \\
\dot{b}_1 &= 8za_1\exp(2zc)\frac{\omega^2+b_2\exp(-2zc_2)}{2z} \\
\dot{c}_1 &= 4a_1
\end{aligned} \tag{0.1}$$

As for the geodesic case, the constants of the motion $\mathcal{H}, \mathcal{C}_z, \delta^{(1)}, \delta^{(2)}$ yield finally first order equations for the above degrees of freedom. The simplest one involves $\exp(2zc_1) = \gamma_1$and reads:

$$\dot{\gamma}_1 = 8z\gamma_1\sqrt{k(\xi_+ - \gamma_1)(\xi_- - \gamma_1)}$$

which is again solvable in terms of trigonometric/hyperbolic functions.

EXAMPLE III

As for the geodesic motion on surfaces with noncostant curvature, just a few preliminary remarks.

Recall that in *polar* variables $\rho, p_\rho, \theta, p_\theta$ for the so-called deformed ADS space-time the Hamiltonian reads:

$$\mathcal{H} = \cos\rho(p_\rho^2 + \frac{p_\theta^2}{\sin^2(\rho)})$$

The corresponding evolution equation for the collective variable $\cos\rho$ is obtained by inverting the elliptic integral:

$$t = \int^{\cos\rho} \frac{dy}{\sqrt{y(E(1-y^2) - p_\theta^2 y)}}$$

In suitable rescaled variables ($a = \cot(p_\theta^2/|E|)$, one get for $\cos\rho$ a periodic motion, with the following period:

$$T = (|E|)^{-\frac{1}{2}} \int_0^a \frac{dx}{\sqrt{x(x-a)(x+a^{-1})}}$$

VII. Classical $\mathcal{U}_q(sl(2))$ Gaudin Hamiltonian and cluster variables

Start now from the *standard* q-algebra , defined by Poisson brackets:

$$\{X_3, X_\pm\} = \pm 2 X_\pm \,, \qquad \{X_+, X_-\} = \frac{\sinh(z X_3)}{z}$$

with Casimir function:

$$\mathcal{C}_z = \frac{\sinh^2\left(\frac{z}{2} X_3\right)}{z^2} + X_+ X_- \,.$$

and admissible coproduct map:

$$\Delta(X_3) = X_3 \otimes I + I \otimes X_3$$
$$\Delta(X_\pm) = X_\pm \otimes e^{\frac{z}{2} X_3} + e^{-\frac{z}{2} X_3} \otimes X_\pm \,.$$

Choosing the function $\mathcal{H} \doteq \Delta^{(N)}(\mathcal{C}_z)$ as our N–body Hamiltonian, the equations of motions are easily written (and solved!) in the alternative basis:

$$S_3 \doteq X_3\,, \qquad S_\pm \doteq e^{-\frac{z}{2}X_3} X_\pm\,,$$

with Poisson brackets

$$\{S_3, S_\pm\} = \pm 2 S_\pm\,, \qquad \{S_+, S_-\} = \frac{1 - e^{-2zS_3}}{2z} + 2z S_+ S_-\,,$$

Casimir

$$\mathcal{C}_z = \frac{\sinh^2\left(\frac{z}{2}S_3\right)}{z^2} + e^{zS_3} S_+ S_-$$

and comultiplication

$$\begin{aligned}\Delta(S_3) &= S_3 \otimes I + I \otimes S_3\\ \Delta(S_\pm) &= S_\pm \otimes I + e^{-zS_3} \otimes S_\pm\,.\end{aligned}$$

They read ($S_i^{(m)} \doteq \Delta^{(m)}(S_i)$):

$$\dot{S}_3^{(m)} = 2e^{z\delta_3}\left(S_+^{(m)}\delta_- - S_-^{(m)}\delta_+\right),$$

$$\dot{S}_\pm^{(m)} = \pm 2z\delta_\mp\, S_\pm^{(m)}\left(\delta_\pm - S_\pm^{(m)}\right) \mp \tfrac{\sinh(zS_3)}{z}\, S_\pm^{(m)} \pm \delta_\pm\, e^{z\delta_3}\tfrac{1-e^{-2zS_3^{(m)}}}{2z}\,,$$

where $\delta_i \doteq \Delta^{(N)}(S_i)$.

VIII. Lax pair for Classical $\mathcal{U}_q(sl(2))$ Gaudin model

To get a Lax pair in the original basis, we make the tentative choice:

$$L^{(m)} \doteq \begin{pmatrix} a^{(m)} & b^{(m)} \\ c^{(m)} & -a^{(m)} \end{pmatrix},$$

where:

$$a^{(m)} = \frac{\sinh\left(\frac{z}{2}X_3^{(m)}\right)}{z},$$
$$b^{(m)} = X_-^{(m)}, \qquad c^{(m)} = X_+^{(m)}.$$

Remark

$$\mathrm{Tr}\left(L^{(m)}\right)^2 = \Delta^{(m)}(\mathcal{C}_z)$$

gives the m–th integral of motion.

Theorem (easy to show): The evolution equation shown in the previous transparency admit the Lax representation:

$$\frac{dL^{(m)}}{dt} = \left\{L^{(m)}, \mathcal{H}\right\} = \left[L^{(m)}, M^{(m,N)}\right],$$

with

$$M^{(m,N)} \doteq \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix},$$

where

$$\alpha = -\frac{\sinh(z\lambda_3)}{z} + 2z\lambda_+\lambda_- - ze^{\frac{z}{2}\lambda_3}e^{-\frac{z}{2}X_3}(c\lambda_- + b\lambda_+)\,,$$
$$\beta = -\frac{1}{2}\lambda_-\left(1 + e^{-zX_3}\right)\, e^{\frac{z}{2}\lambda_3}\,,$$
$$\gamma = -\frac{1}{2}\lambda_+\left(1 + e^{-zX_3}\right)\, e^{\frac{z}{2}\lambda_3}\,.$$

and

$$\lambda_3 \doteq \Delta^{(N)}(X_3), \qquad \lambda_\pm \doteq \Delta^{(N)}(X_\pm)$$

.

The (m) superscript has been omitted.

So far, we have a Lax pair for each cluster variable. A global Lax pair can be obtained by adding together all partial $L^{(m)}$ matrices in block–diagonal form:

$$L(\lambda) == \bigoplus_{j=1}^{N} \lambda^{1-j} L^{(j)}\,, \qquad M = \bigoplus_{j=1}^{N} M^{(j)}\,.$$

The generating function of the integrals of motion is

$$\mathrm{Tr} L(\lambda)^2 = \sum_{j=1}^{N-1} \lambda^{2\,(1-j)} \Delta^{(j)}(\mathcal{C}_z)\ ,.$$

Perspectives and Open Problems: some examples

- Part I : The extension to N degrees of freedom is a work in progress in the classical case; however, very little has been done in the Quantum context (see Kalnins, Miller, Winternitz among others)
- Part II: A global r-matrix structure has to be found (preliminary encouraging results). The Lax representation has to be extended to the Quantum case. And finally: do exist integrable models of Gaudin type (hence, long-range) which combine $q-$symmetry and inhomogeneities?