Noncompact Heisenberg spin chains induced by Yang-Mills theories

Gregory Korchemsky
LPT, Orsay

in collaboration with:

Andrei Belitsky (Arizona State U.),
Vladimir Braun (Regensburg U.),
Sergey Derkachov (St.Petersburg U.),
Alexander Gorsky (ITEP, Moscow),
Alexander Manashov (Regensburg U.),
Dieter Müller (Regensburg U.)

For more details see review hep-th/0407232

What are the symmetries of Yang-Mills theories?

- ✔ QCD = (3+1)-dimensional Yang-Mills field theory with the $SU(N_c = 3)$ gauge group

$$S_{\rm QCD} = \int d^4x \left[-\frac{1}{4} F^2_{\mu\nu}(x) + \sum_{q=u,d,\dots} \bar{q}(i\not{D} - m_q)q(x) \right]$$

- ✔ Symmetry of the *classical* theory:
  - ✗ gauge symmetry: $q(x) \to e^{i\omega} q(x), \quad F_{\mu\nu}(x) \to e^{i\omega} F_{\mu\nu}(x) e^{-i\omega}$
  - ✗ chiral symmetry (for $m_q = 0$): $q(x) \to e^{i\gamma_5\omega} q(x), \quad F_{\mu\nu}(x) \to e^{i\omega} F_{\mu\nu}(x) e^{-i\omega}$
  - ✗ conformal symmetry (dilatations): $q(x) \to \lambda^{d_q} q(\lambda x)\,, \quad F_{\mu\nu}(x) \to \lambda^{d_g} F_{\mu\nu}(x)$
- ✔ Most of classical symmetries are broken on the *quantum* level:
  - ✗ gauge symmetry is protected: $\partial^\mu J_\mu = 0$
  - ✗ chiral anomaly: $\partial^\mu J^5_\mu \neq 0$
  - ✗ conformal anomaly: $\partial^\mu J^D_\mu = (\beta_{\rm QCD}(g)/g)\; F^2_{\mu\nu}(x)$

Q: *Could it be that QCD possesses a hidden symmetry which does not exhibit itself as a symmetry of the classical Lagrangian but is only revealed on the quantum level?*

Hidden symmetry of QCD

A: *Yes!* QCD at high energy is intrinsically related to *completely integrable models*

- ✔ Integrable models = QM systems with a *finite* number of degrees of freedom and the same number of conserved charges.
- ✔ Gauge theories in four dimensions = complex systems with *infinite* number of degrees of freedom which are not integrable *per se*.
- ✔ Integrability emerges as a hidden symmetry of *effective* Yang-Mills dynamics in two *different* limits:
  - ✗ High-energy (Regge) behaviour of scattering amplitudes in QCD

$$\mathcal{A}_{\rm BFKL}(s) \sim s^{E}\,, \qquad\qquad E = \frac{g^2 N_c}{8\pi^2}{\rm Re}[\psi(J) - \psi(1)]$$

  - ✗ Anomalous dimensions of composite (Wilson) operators in QCD

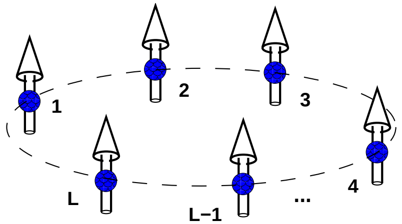
$$\mathcal{O}_N(0) = \bar{q}^{\uparrow}(0)\gamma_{\{\mu_1}D_{\mu_2}\dots D_{\mu_N\}}q^{\uparrow}(0) \sim (\Lambda_{\rm UV})^{-\gamma_N}\,, \qquad \gamma_N = \frac{g^2 N_c}{8\pi^2}[\psi(N+2) - \psi(1)]$$

- ✔ $\psi(x) = d\ln\Gamma(x)/dx$ is not just a function ... but indication of hidden integrability (= Heisenberg spin chains)
- ✔ Integrability is not tied to QCD and is a general feature of (super) YM in four dimensions

Heisenberg spin chains

- One-dimensional chain of atoms with exchange interaction *Heisenberg'26*

$$\mathbb{H}_{s=1/2} = \sum_{n=1}^{L} \left(\mathbf{S}_n \cdot \mathbf{S}_{n+1} - \frac{1}{4} \right) =$$



- The model is completely integrable and can be solved exactly *Bethe'31*

- It can be generalized to arbitrary $SU(2)$ and $SL(2)$ spins $\mathbf{S}_n^2 = s(s+1)$ while preserving integrability

Kulish,Reshetikhin,Sklyanin'81
Faddeev,Tarasov,Takhtajan'83

$$\mathbb{H}_s = \sum_{n=1}^{L} H(J_{n,n+1})$$

  - $J_{n,n+1}$ = the sum of two neighboring spins, $J_{n,n+1}(J_{n,n+1}+1) = (\mathbf{S}_n + \mathbf{S}_{n+1})^2$
  - Two-particle Hamiltonian = the Euler $\psi-$function, harmonic sum

$$H(x) = -\sum_{l=x}^{2s-1} \frac{1}{l+1} = \psi(x+1) - \psi(2s+1)$$

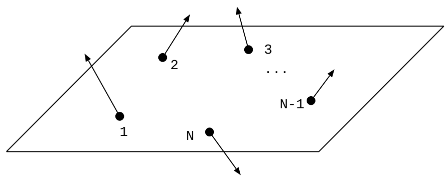
- Integrable structures in high-energy QCD:
  - QCD in the Regge limit $\Longrightarrow SL(2,\mathbb{C})$ spin chain *Lipatov'93; Faddeev, GK'94*
  - QCD on the light-cone $\Longrightarrow SL(2,\mathbb{R})$ spin chain *Braun,Derkachov,Manashov'98; Belitsky; GK'99*

The $SL(2)$ Heisenberg spin chain induced by QCD

- QM system of N interacting particles “living” on a two-dimensional plane in Minkowski space-time

$$x_\mu x^\mu = \underbrace{t^2 - \mathrm{z}^2}_{\text{Light}-\text{cone}} - \underbrace{\mathrm{x}^2 - \mathrm{y}^2}_{\text{Regge limit}} = z^+ z^- - z\bar{z}$$

- The position of the particles on the plane is defined by light-cone/holomoprhic coordinates



$$z_k = \begin{cases} t_k + \mathrm{z}_k & \leftarrow \text{Light} - \text{cone}, \\ \mathrm{x}_k + i\mathrm{y}_k & \leftarrow \text{Regge limit}, \end{cases} \qquad (k = 1, ..., N)$$

- Each particle carries the $SL(2)$ spin

$$S_k^0 = z_k \partial_{z_k} + s\,, \quad S_k^- = -\partial_{z_k}\,, \quad S_k^+ = z_k^2 \partial_{z_k} + 2s z_k\,,$$

- The effective QCD Hamiltonian describes interaction between noncompact $SL(2)$ spins attached to N particles

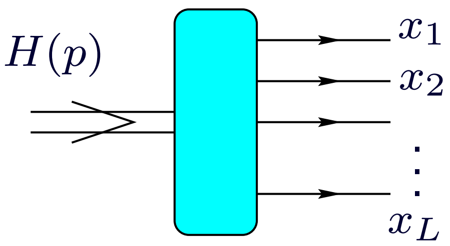
$$\mathbb{H}_{\text{QCD}} = \sum_{k=1}^{N} H(J_{k,k+1})\,, \qquad H(x) = \psi(x+1) - \psi(2s+1)$$

$J_{k,k+1} =$ the sum of two spins, $(\vec{S}_k + \vec{S}_{k+1})^2 = J_{k,k+1}(J_{k,k+1} - 1)$.

- The model is completely integrable: $[q_k, \mathbb{H}_{\text{QCD}}] = [q_k, q_n] = 0$

Multi-particle operators in QCD on the light-cone

- ✔ Feynman-Gribov parton model: hadrons in the infinite momentum frame ≈ system of quasi-free partons



$$\underbrace{0 \le x_k \le 1\,, \quad \sum_k x_k = 1}_{\text{momentum fractions}}$$

- ✔ Distribution (baryon) amplitude: *Brodsky, Lepage'79*

$$\langle 0|q(z_1 n)q(z_2 n)q(z_3 n)|H(p)\rangle \overset{n_\mu^2=0}{=} \int_0^1 dx_1 dx_2 dx_3\, \delta(\sum x_k - 1)\, e^{-i(pn)\sum_k x_k z_k} \varphi_B(x_i,\mu^2)$$

Nonlocal light-cone correlator = sum of plane waves

☞ *How to find the scale dependence of the distribution amplitude $\varphi_B(x_i,\mu^2)$?*

- ✔ Moments of distribution amplitudes $\Longleftrightarrow$ local operators:

$$\varphi_B(x_i) \to \widetilde{\varphi}_B(k_i) \quad = \quad \int \mathcal{D}x\, x_1^{k_1} x_2^{k_2} x_3^{k_3}\, \varphi_B(x_i,\mu^2) = \langle 0|(D_+^{k_1} q)\,(D_+^{k_2} q)\,(D_+^{k_3} q)|H(p)\rangle$$

- ✔ Scale dependence of the moments

$$\mu\frac{d}{d\mu}\widetilde{\varphi}_B(k_i) = \sum_{m_j} \underbrace{V(k_i|m_j)}_{\text{mixing matrix}}\ \widetilde{\varphi}_B(m_j)$$

Conventional QCD approach

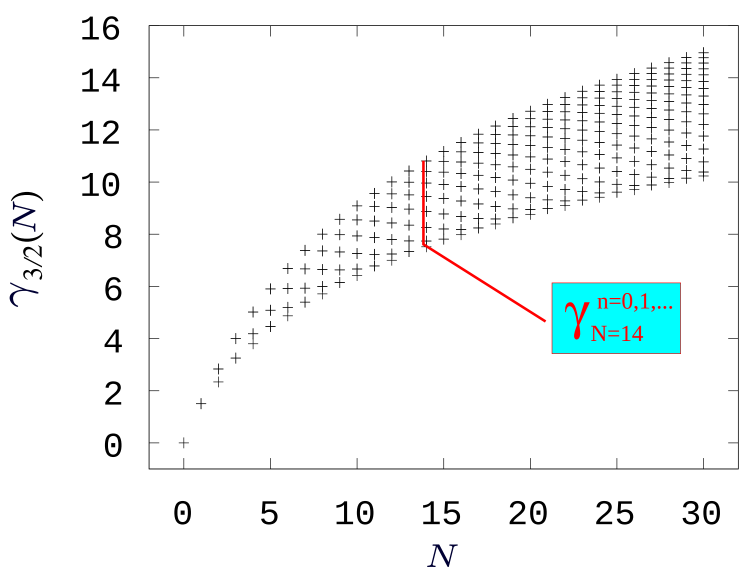
- ✔ Diagonalize the mixing matrix and find the spectrum of anomalous dimensions
- ✔ Example: helicity$-3/2$ baryon distribution amplitude $[q = q^{\uparrow}(x) + q^{\downarrow}(x),\ q^{\uparrow(\downarrow)} = \frac{1\pm\gamma_5}{2} q]$

$$q^{\uparrow}(z_1 n) q^{\uparrow}(z_2 n) q^{\uparrow}(z_3 n) \longrightarrow (D_+^{k_1} q^{\uparrow})\,(D_+^{k_2} q^{\uparrow})\, q^{\uparrow}(0) + [\text{total derivatives}]$$

  - ✗ Mixing matrix:

$$\sum_{n_1+n_2=N} V(k_1, k_2 | n_1, n_2)\, C^{(\ell)}_{n_1,n_2} = \gamma^{(\ell)}_{3/2}(N)\, C^{(\ell)}_{k_1,k_2}\,, \qquad (\ell = 0, \dots, N)$$

  - ✗ Rich spectrum of anomalous dimensions:



- (Almost) all levels are double degenerate
- Where does this structure come from?

  Conformal symmetry + Integrability!

Conformal symmetry on the light-cone

- ✔ QCD Lagrangian is invariant under the $SO(4,2)$ transformations:

$$\underbrace{x_\mu \to x_\mu + a_\mu\,,}_{\text{translations}} \qquad \underbrace{x_\mu \to \omega_{\mu\nu} x_\nu\,,}_{\text{rotations}} \qquad \underbrace{x_\mu \to \lambda x_\mu\,,}_{\text{dilatations}} \qquad \underbrace{x_\mu \to c_\mu (x^2 g_{\mu\nu} - x_\mu x_\nu)}_{\text{conformal boosts}}$$

- ✔ $SO(4,2)$ reduces on the light-cone $x_\mu = z n_\mu$ ($n^2 = 0$) to the $SL(2)$ subgroup: *Ohrndorf'83*

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad q(z) \to q'(z) = q\left(\frac{az+b}{cz+d}\right) \cdot (cz+d)^{-2j_q}$$

"z" - position of quark field on the light-cone

  - ✗ The $SL(2)$ generators:

$$S_- = -\frac{d}{dz}\,, \quad S_+ = \left(z^2\frac{d}{dz} + 2zj_q\right)\,, \quad S_0 = \left(z\frac{d}{dz} + j_q\right)$$

  - ✗ $j_q = 1$ is the *conformal spin* of the quark field

- ✔ Conformal symmetry is broken in QCD ($\beta_{\rm QCD}(g) \neq 0$) but the conformal anomaly affects the anomalous dimensions starting from two loops only

- ✔ One-loop evolution kernels in QCD inherit conformal symmetry of the classical Lagrangian! *Zamolodchikov*

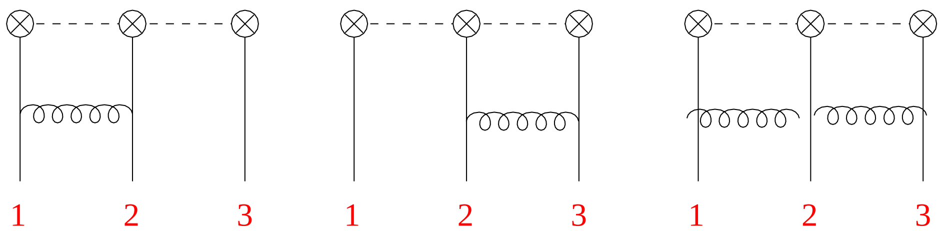
Integrability on the light-cone

- Callan-Symanzik equation (helicity$-\frac{3}{2}$ baryon operator $B \equiv q^{\uparrow}(z_1 n) q^{\uparrow}(z_2 n) q^{\uparrow}(z_3 n)$)

$$\mu \frac{d}{d\mu} B(z_1, z_2, z_3) = [\mathbb{H} \cdot B](z_1, z_2, z_3),$$

- Two-particle structure of one-loop dilatation operator:

$\mathbb{H}_{1-\text{loop}} =$



$= \frac{g_s^2 N_c}{8\pi^2} [\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{13}]$

- Two-particle kernel: *Balitsky, Braun'88*

$$\mathcal{H}_{12} B(z_1, z_2, z_3) = \int_0^1 \frac{d\alpha\, \alpha}{1-\alpha} \Big[2B(z_1, z_2, z_3) - B(\alpha z_1 + (1-\alpha) z_2, z_2, z_3) - B(z_1, \alpha z_2 + (1-\alpha) z_1, z_3) \Big]$$

  - displaces quark fields along the light-cone
  - conformal invariance: $[\mathbb{H}, \vec{S}_1 + \vec{S}_2 + \vec{S}_3] = 0$
  - dependence on the two-particle conformal spin $J_{12} = N + 2$ [with $B(z_i) \to (z_1 - z_2)^N$]

$$\mathcal{H}_{12} = \int_0^1 \frac{d\alpha\, \alpha}{1-\alpha} [2 - 2\alpha^N] = 2 [\psi(J_{12}) - \psi(2)]$$

Integrability on the light-cone (II)

- ✔ QCD anomalous dimensions are eigenvalues of the dilatation operator

$$\mathbb{H}\,\Psi_N(z_1,z_2,z_3) = \gamma_N\,\Psi_N(z_1,z_2,z_3)$$

- ✔ $SL(2)$ invariant form of the dilatation operator

$$\mathbb{H} = \frac{g_s^2 N_c}{8\pi^2}\left[\mathcal{H}_{12}+\mathcal{H}_{23}+\mathcal{H}_{13}\right], \qquad \mathcal{H}_{jk} = 2\Big[\psi(J_{jk}) - \psi(2)\Big]$$

Two-particle conformal spin

$$(\vec{S}_j + \vec{S}_k)^2 \equiv J_{jk}(J_{jk}-1)$$

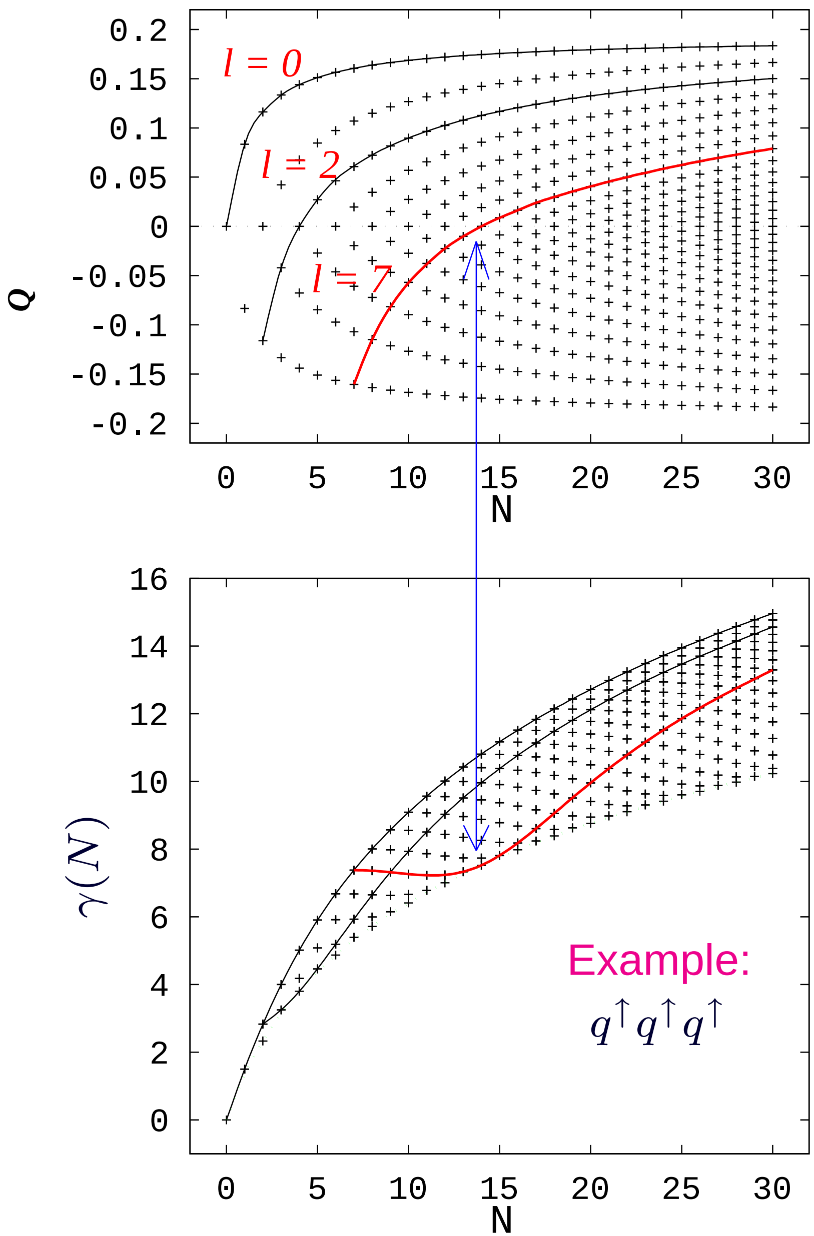
- ✔ One-loop dilatation operator $\equiv$ Hamiltonian of the $SL(2,\mathbb{R})$ Heisenberg spin chain
  - ✗ Number of sites = number of quark operators
  - ✗ Spin operators = Generators of the $SL(2,\mathbb{R})$ 'collinear' group
  - ✗ Quantum space = discrete series of the $SL(2;\mathbb{R})$ group

Braun, Derkachov, Manashov'98
Belitsky; GK'99

$$\Psi_N(z_1,z_2,z_3) = \sum_{k_1,k_2,k_3} z_1^{k_1} z_2^{k_2} z_3^{k_3} C_{k_1 k_2 k_3} = \textit{polynomials}$$

  - ✗ Possesses the lowest weight $\Psi_N^{(0)} = 1$ (with $S^- \cdot \Psi_N^{(0)} = 0$) but not the highest weight
  - ✗ The spectrum of anomalous dimensions can be found exactly by means of the Algebraic Bethe Ansatz

(Exact) Bethe Ansatz solution



✔ Bethe equations

$$\gamma_N = \frac{g_s^2 N_c}{8\pi^2} \sum_{k=1}^{N} \frac{2j_q}{\lambda_k^2 + j_q^2},$$

$\{\lambda_1, \ldots, \lambda_N\} =$ Bethe roots;
$j_q = 1$ conformal spin of quark

$$\left(\frac{\lambda_k + ij_q}{\lambda_k - ij_q}\right)^3 = \prod_{j\neq k}^{N} \frac{\lambda_k - \lambda_j - i}{\lambda_k - \lambda_j + i}$$

✔ (Almost) all levels are double degenerate:

$$\gamma(N,Q) = \gamma(N,-Q)\,, \qquad Q(N,\ell) = -Q(N,N-\ell)$$

$N =$total spin, $\quad Q =$conserved charge

✔ Integrability imposes a nontrivial analytic structure – trajectories are enumerated by integer $\ell = 0, 1, \ldots$

Separated variables/Baxter $\mathbb{Q}-$operator

Go over to the representation of Separated Variables *Sklyanin'85*

$$\Psi_q(z_1,...,z_L) \mapsto \Psi_q^{(\mathrm{SoV})} = \mathbb{U}\,\Psi_q(z_1,...,z_L)$$

Explicit construction of the SoV representation for the $SL(2)$ magnet *Derkachov, GK, Manashov'02*

$$\Psi_q^{(\mathrm{SoV})} = Q(x_1)\dots Q(x_{L-1})\,, \qquad \gamma_q = i\frac{Q'(ij_q)}{Q(ij_q)} - i\frac{Q'(-ij_q)}{Q(-ij_q)}$$

$Q(u) =$ eigenvalue of the Baxter operator

$$\mathbb{Q}(u)\Psi_q(z_i) = Q(u)\Psi_q(z_i)$$

$t-Q$ relations:

$$[t_L(u),\mathbb{Q}(u)] = [\mathbb{Q}(u),\mathbb{Q}(v)] = 0$$

$$(u+ij)^L\mathbb{Q}(u+i) + (u-ij)^L\mathbb{Q}(u-i) = t_L(u)\,\mathbb{Q}(u)$$

$t_L(u) = 2u^L + q_2u^{L-2} + \ldots + q_L$

Exact solution (equivalence with the Algebraic Bethe Ansatz)

$$Q(u) = \prod_{k=1}^{N}(u-\lambda_k) = \textit{polynomial} \text{ in } u$$

$\{\lambda_1,...,\lambda_N\}$ =Bethe roots; $\qquad N =$ total conformal spin

Integrable sectors in multi-color QCD on the light-cone

✔ Interaction between partons with the *aligned* helicities (quarks $q^\uparrow$, gluons $G^\uparrow$) is integrable

One-loop dilatation operator $\mathbb{H}$ = Hamiltonian of a noncompact $SL(2,\mathbb{R})$ Heisenberg magnet:

Braun, Derkachov, Manashov; Belitsky; GK'99

- ✗ Three-quark states:

$$[q^\uparrow(z_1)q^\uparrow(z_2)q^\uparrow(z_3)] \Longrightarrow \text{ closed spin } j_q = 1 \text{ chain}$$

- ✗ Multi-gluon states:

$$[G^\uparrow(z_1)G^\uparrow(z_2)...G^\uparrow(z_L)] \Longrightarrow \text{ closed spin } j_g = 3/2 \text{ chain}$$

- ✗ Antiquark-Glue-Quark states:

$$[\bar{q}(z_1)\, G^\uparrow(z_2)...G^\uparrow(z_{L-1})q(z_L)] \Longrightarrow \text{ open inhomogeneous spin chain}$$

✔ Integrability is broken in the 'mixed' helicity sectors (ex: helicity$-1/2$ states $[q^\uparrow q^\downarrow q^\uparrow]$)

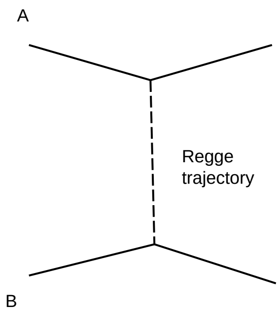
- ✗ Symmetry breaking terms generate a mass gap in the spectrum of $\gamma(N)$ [scalar diquarks]

✔ Supersymmetry enhances QCD integrability and extends it to a larger class of Wilson operators as one goes from $\mathcal{N}=0$ to $\mathcal{N}=4$ SYM

Scattering in QCD at high-energy (Regge asymptotics)

✔ Regge phenomena in strong interactions (since 60's):

$$\sigma_{\rm AB}(s) =$$

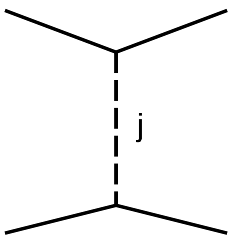


$$= \sum_j \beta_A^j(t)\beta_B^j(t)\, s^{\alpha_j(t)-1}$$

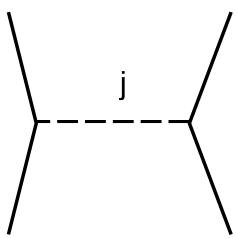
Scattering amplitudes grow at high energy s as a power $\sim s^{\alpha_j(t)}$

✔ Dual model:

$\sum_j$



$= \sum_j$



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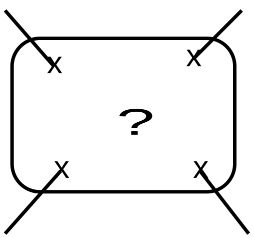


Regge trajectories + duality condition = Hadronic string (?)

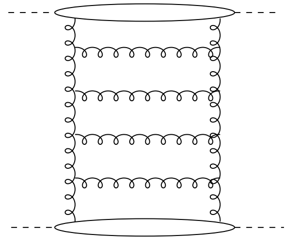
✔ High-energy asymptotics in QCD: interaction induces large corrections which need to be resummed to all order of perturbation theory *Balitsky-Fadin-Kuraev-Lipatov '78*:

$$\sigma_{\rm AB}(s) = \sum_{n=0,1,\dots} w_n \left(g_s^2 \ln s\right)^n \sim s^{\alpha_{\mathbb{P}}-1}$$

BFKL Pomeron + Unitarity

- Leading contribution: BFKL Pomeron ($\lambda = g_s^2 N_c/(4\pi^2)$)

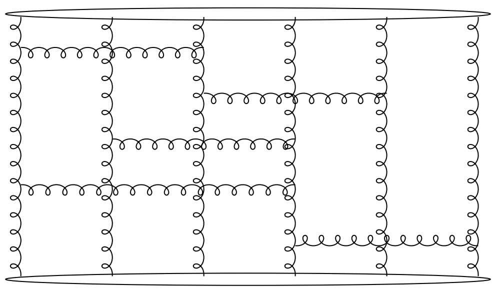
$$\sigma_{\rm LO} = \sum_{\rm rungs}$$



$$\sim \lambda^2 \frac{\exp(4\ln 2 \cdot \lambda \ln s)}{\sqrt{\lambda \ln s}} \sim \underbrace{s^{4\ln 2 \cdot \lambda}}_{\text{violates unitarity}}$$

- BFKL Pomeron + Unitarity $\Rightarrow$ generalized ladder diagrams

$$\sum_{N=2,3,..}$$



- Multi-Regge kinematics: $\int d^4k = \int dk_+ dk_- \int d^2k_\perp$

– strong ordering in the longitudinal momenta $y = \ln \frac{k_+}{k_-}$

$$y_1 \gg y_2 \gg y_3 \gg ... = \text{“evolution time” in the t} - \text{channel}$$

– “random walk” in the transverse momenta

$$k_{1,\perp} \sim k_{2,\perp} \sim k_{3,\perp} \sim ...$$

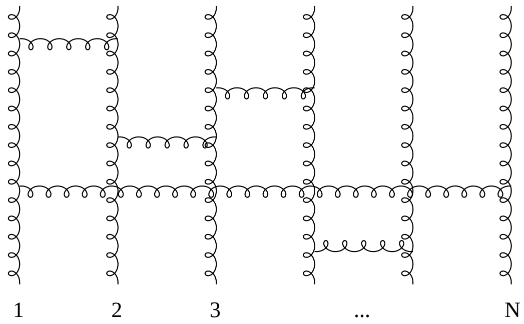
✌ Elastic pair-wise interaction of $N = 2, 3, ...$ particles “living” on the two-dimensional $k_\perp$ –plane and propagating in the “time” $y = \ln s$.

✌ Nontrivial QCD dynamics occurs on the two-dimensional transverse space

Color-singlet compound gluonic states

☞ The effective QCD Hamiltonian $\mathcal{H}_N$ has remarkable properties in the multi-color limit:

$\sigma(s) =$



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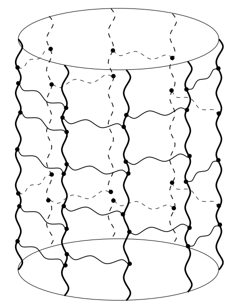


$\sim \exp(y\mathcal{H}_N) = s^{\mathcal{H}_N}$

Elastic scattering of N particles

✔ The Bartels-Kwiecinski-Praszalowicz equation $\equiv$ 2-dim Schrödinger equation

$$\mathcal{H}_N \Psi\underbrace{(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)}_{2-\text{dim coordinates}} = E_N \Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)$$

✔ $\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N) =$ colour-singlet compound states built from N reggeized gluons

✔ Interaction occurs only between nearest neighbours

$$\mathcal{H}_N = \sum_{k=1}^{N} \underbrace{H(\vec{z}_k, \vec{z}_{k+1})}_{\text{BFKL kernel}} + \mathcal{O}(1/N_c^2) = N-\textit{body QM system with periodic boundary conditions}$$

✔ High-energy asymptotics of the scattering amplitudes governed by these states

$$\sigma(s) \sim \sum_{\substack{N-\text{gluon}\\ \text{states}}} (i\lambda)^N \underbrace{s^{\lambda E_N}}_{\text{Regge behaviour}} \beta_N(t)$$

✔ Intercept = maximal energy E_N

Integrability

- ✔ The system possesses a "hidden" set of the integrals of motion *Lipatov'93; Faddeev, GK'94*

$$[q_k, \mathcal{H}_N] = [q_k, q_n] = 0\,, \qquad (k, n = 2, 3, \ldots, N)$$

Their number is large enough $(= N)$ for the Schrödinger equation to be *completely integrable !*

- ✔ The system of N reggeized gluons $\equiv$ completely integrable quantum-mechanical model
  - ✗ $\mathcal{H}_N \xlongequal{N_c\to\infty} SL(2,\mathbb{C})$ Heisenberg spin chain on 2-dim plane $ds^2 = dzd\bar{z}$

$$\mathcal{H}_N = SL(2;\mathbb{R}) + \overline{SL(2;\mathbb{R})} = \text{[holomorphic] + [anti-holomorphic]}$$

  - ✗ Number of sites = number of reggeized gluons
  - ✗ Spin operators = generators of (infinite-dimensional) irreps of the $SL(2,\mathbb{C})$ group

$$J_{\rm tot} = \frac{1+n}{2} + i\nu\,, \qquad \bar{J}_{\rm tot} = \frac{1-n}{2} + i\nu\,, \qquad (n \in \mathbb{N}_+\,, \nu \in \mathbb{R})$$

  - ✗ Eigenstates = functions well-defined on the *plane* (but not in z and $\bar{z}$ separately!)

$$\Psi(\vec{z}_1, \ldots, \vec{z}_N) = \sum C_{ab} \Psi^{(a)}(z_1, \ldots, z_N) \overline{\Psi^{(b)}}(\bar{z}_1, \ldots, \bar{z}_N)$$

- ✔ Can the spectrum of the QCD Hamiltonian $\mathcal{H}_N$ be found exactly? *Derkachov, GK, Manashov'01*
  - ✗ The $SL(2,\mathbb{C})$ magnet does not contain a pseudovacuum state (= highest weight)
  - ✗ ... the "conventional" Bethe Ansatz is *not* applicable
  - ✗ The way out: Baxter $\mathbb{Q}-$operator + Sklyanin's SoV approach

Exact solution of the $SL(2;C)$ spin chain

- The eigenstates of the $SL(2;\mathbb{C})$ in the SoV representation *Derkachov, GK, Manashov'02*

$$\Psi_q(\vec{z}_1,...,\vec{z}_L) \mapsto \Psi_q^{(\rm SoV)} = \mathbb{U}\,\Psi_q(\vec{z}_1,...,\vec{z}_L) = Q(x_1,\bar{x}_1)\ldots Q(x_{L-1},\bar{x}_{L-1})$$

$(x_k,\bar{x}_k) =$ separated variables, $\quad Q(x_k,x_k) =$ eigenvalue of the Baxter operator

$$\mathbb{Q}(u,\bar{u})\Psi_q(\vec{z}_1,...,\vec{z}_L) = Q(u,\bar{u})\Psi_q(\vec{z}_1,...,\vec{z}_L)$$

- Energy spectrum of the spin chain $(s=0,\bar{s}=1)$

$$E_q = i\frac{d}{du}\ln\left[Q(u+is,u+i\bar{s})\left(Q(u-is,u-i\bar{s})\right)^*\right]\Big|_{u=0}$$

- Holomorphic $t-Q$ relations:

$$[t_L(u),\mathbb{Q}(u,\bar{u})] = [\mathbb{Q}(u,\bar{u}),\mathbb{Q}(v,\bar{v})] = 0$$

$$(u+is)^L\mathbb{Q}(u+i,\bar{u}) + (u-is)^L\mathbb{Q}(u-i,\bar{u}) = t_L(u)\,\mathbb{Q}(u,\bar{u})$$

$t_L(u) = 2u^L + q_2u^{L-2} + \ldots + q_L$

The *same* as for the $SL(2;\mathbb{R})$ spin chain but $\mathbb{Q}(u,\bar{u})$ is *not* polynomial in u and $\bar{u}$

- Analytical properties

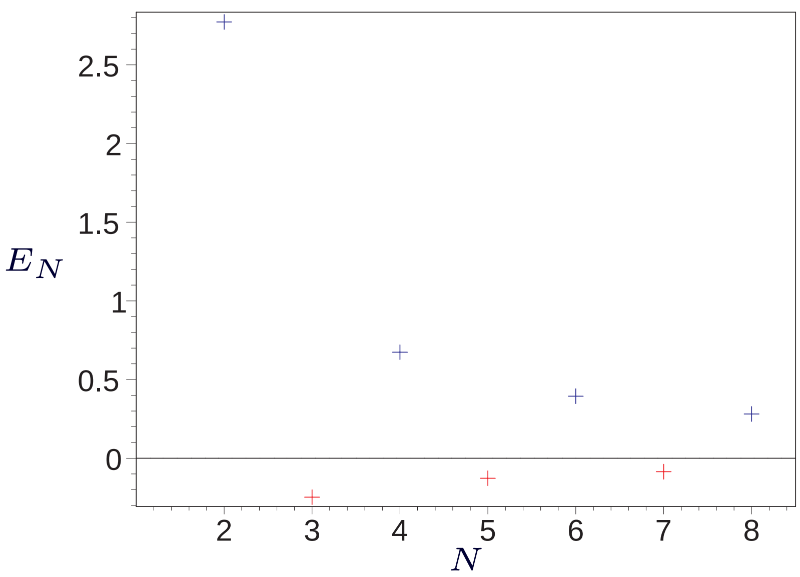
$$Q(\lambda - in/2, \lambda + in/2) \quad = \quad$$ *meromorphic* function of λ with prescribed position/order of poles

Energy spectrum

- ✔ Quantum numbers, $q_2, ..., q_N$, and the energy, E_N, of $N-$reggeon states in multi-color QCD:
  Derkachov, GK, Kotanski, Manashov' 01

N	q_2	iq_3	q_4	iq_5	q_6	E_N
2	.25					2.77259
3	.25	.20526				-.24717
4	.25	0	.15359			.67416
5	.25	.26768	.03945	.06024		-.12751
6	.25	0	.28182	0	.07049	.39458

- ✔ The ground state energy of the $SL(2;\mathbb{C})$ spin chain of length N



- ✗ Different scaling behaviour for even and odd N
- ✗ Thermodynamical limit

$$E_N \to 0 \qquad \text{for} \qquad N \to \infty$$

needs to be understood...

Conclusions

- ✔ High-energy QCD possesses a hidden symmetry in two *different* limits:
  - ✗ Light-cone
  - ✗ Regge limit
- ✔ In both cases one encounters the *same* integrable structure $\equiv$ XXX Heisenberg noncompact spin chain
- ✔ The symmetry is powerful enough to calculate using the Bethe Ansatz
  - ✗ the exact spectrum of anomalous dimensions
  - ✗ intercepts of the compound states in multi-color QCD
- ✔ ... *but we do not quite understand the origin of integrability in gauge theories*