The correlation functions of the XYZ chain

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Recent work in collaboration with

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- Algebraic Bethe ansatz was constructed by Faddeev and Takhtadjan in 1979
- The XYZ Hamiltonian for the infinite chain looks:

$$H_{XYZ} = \sum_{j=-\infty}^{\infty} \left(I_x \sigma_j^x \sigma_{j+1}^x + I_y \sigma_j^y \sigma_{j+1}^y + I_z \sigma_j^z \sigma_{j+1}^z \right)$$

and corresponds to the 8-vertex model with R-matrix:

$$R(t) := \rho(t)\frac{r(t)}{[t+\eta]} \qquad \in \mathrm{End}(V \otimes V),$$

$$r(t) := \frac{1}{2}\sum_{\alpha=0}^{3} \frac{\theta_{\alpha+1}(2t+\eta)}{\theta_{\alpha+1}(\eta)}\sigma^{\alpha} \otimes \sigma^{\alpha},$$

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where $\theta_\alpha(t) = \theta_\alpha(t|\tau)$ $(0\le\alpha\le 4,\ \theta_4(t)=\theta_0(t))$ denote the Jacobi elliptic theta functions associated with the lattice $\mathbb{Z}+\mathbb{Z}\tau$, $\quad (\mathrm{Im}\,\tau>0,\ \eta\notin\mathbb{Q}+\mathbb{Q}\tau)\quad$ and

$$[t] := \frac{\theta_1(2t)}{\theta_1(2\eta)}, \qquad \rho(t)\rho(-t)=1, \quad \rho(t)\rho(t-\eta)=\frac{[t]}{[\eta-t]}$$

Introduction — The XYZ chain

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$$\Gamma(u, \sigma, \tau) := \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - u)}}{1 - e^{2\pi i(j\tau + k\sigma + u)}}$$

is the elliptic gamma function.

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$$\rho^{ord}(t) = e^{-4\pi i \eta t/\tau} \rho'(t'),$$

where $\rho'(t')$ has the same form as ρ^{dis} but with t, η, τ being replaced by

$$t' = \frac{t}{\tau}, \quad \eta' = \frac{\eta}{\tau}, \quad \tau' = -\frac{1}{\tau},$$

respectively.

Correlation functions and the reduced qKZ

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- dynamic (time-dependent) correlation functions
- thermal correlators
- time-independent correlation functions at zero temperature

Below we shall be dealing only with the last case.

Correlation functions and the reduced qKZ

Any correlation function can be reduced to a linear combination of so-called elementary blocks or elements of the density matrix

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$$P^{\bar\varepsilon_1,\ldots,\bar\varepsilon_n}_{\varepsilon_1,\ldots,\varepsilon_n} := \langle \mathrm{vac}|E^{\bar\varepsilon_1}_{\varepsilon_1}\ldots E^{\bar\varepsilon_n}_{\varepsilon_n}|\mathrm{vac}\rangle$$

$$(E^{\bar\varepsilon}_{\varepsilon})_{s,s'} := \delta_{\varepsilon,s}\delta_{\bar\varepsilon,s'} \quad \varepsilon,\bar\varepsilon = \pm 1$$

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In the inhomogeneous case vacuum depends on the spectral parameters: $\quad |\mathrm{vac}\rangle \equiv |\mathrm{vac};\lambda_1,\cdots,\lambda_n\rangle$

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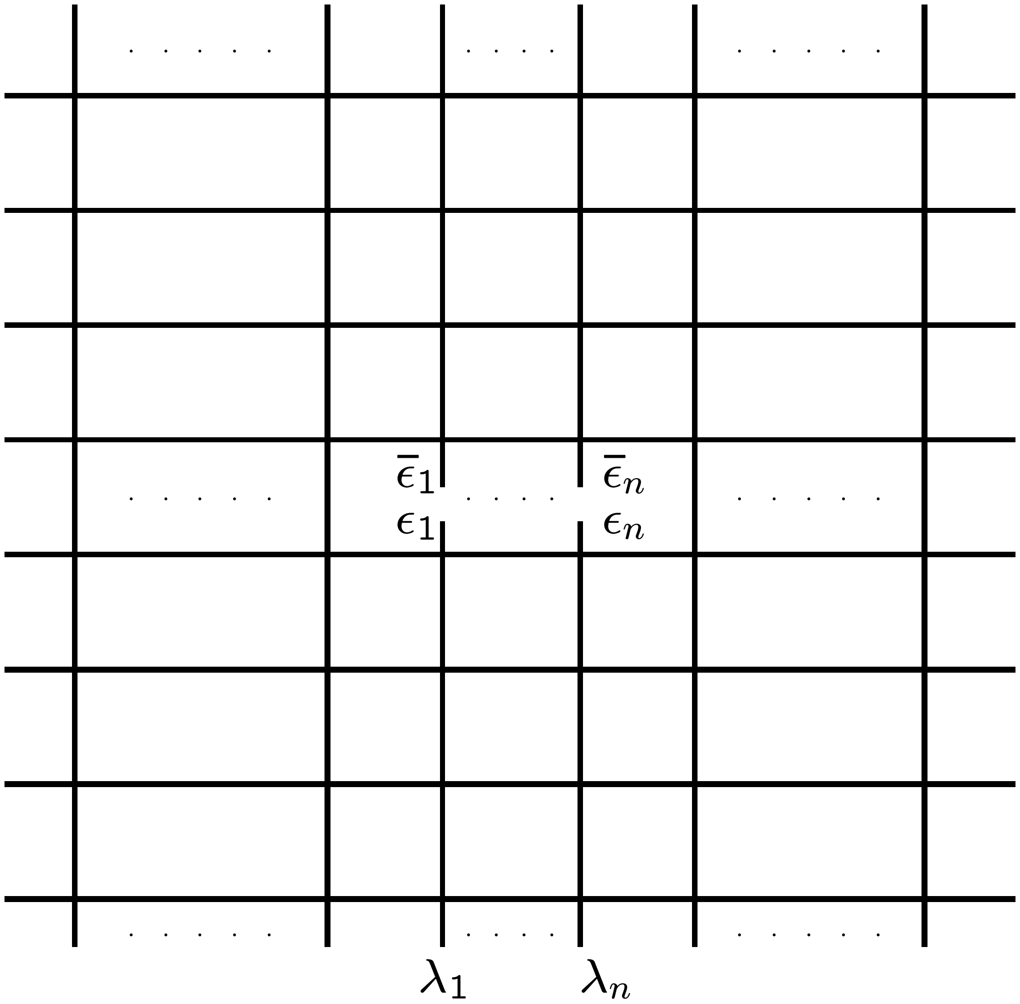
$$P^{\bar\varepsilon_1,\ldots,\bar\varepsilon_n}_{\varepsilon_1,\ldots,\varepsilon_n} \equiv P^{\bar\varepsilon_1,\ldots,\bar\varepsilon_n}_{\varepsilon_1,\ldots,\varepsilon_n}(\lambda_1,\cdots,\lambda_n)$$

Correlation functions and the reduced qKZ



Infinite square lattice

Short survey of some results

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- Staggered polarization was calculated by Baxter, 73-74 using the corner transfer matrix method
- A basic role in the study of correlation functions is played by a multiple integral representation, first found for the spin $1/2$-XXZ chain by Jimbo, Miki, Miwa, Nakayashiki, (92) using the vertex operator approach and then reproduced by Kitanine, Maillet, Terras, Slavnov, (1998-2000) in framework of the algebraic Bethe ansatz approach

Correlation functions and the reduced qKZ

- The free field construction used in the XXZ model was extended by Lukyanov and Pugai, (96) to the SOS models, resulting in an integral formula for correlation functions of the ABF model.

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- Then Lashkevich and Pugai, (98) obtained an integral formula for the eight-vertex model by mapping the problem to the SOS counterpart.
- A novel free field representation of the eight-vertex model is being developed by Shiraishi, (2004-2005)

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It was observed by Jimbo and Miwa, (96) that the correlation functions are related to special solutions of the quantum Knizhnik-Zamolodchikov equation [qKZ].

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From this relation follows that the vector

$$h_n(t_1,\ldots,t_n)_{\varepsilon_1\ldots\varepsilon_n\bar{\varepsilon}_n\ldots\bar{\varepsilon}_1} := \prod_{j=1}^{n}(-\bar{\varepsilon}_j)P_{-\varepsilon_1,\ldots,-\varepsilon_n}^{\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_n}(\lambda_1,\ldots,\lambda_n),$$

$$t_j := \lambda_j\eta$$

which takes values in $V^{\otimes 2n}$ (sometimes we write it $h_n(t_1,\ldots,t_n)_{1,\cdots,n,\bar{n},\cdots,\bar{1}}$) satisfies the so-called reduced qKZ [rqKZ]

Correlation functions and the reduced qKZ

(1) $h_n(\cdots,t_{j+1},t_j,\cdots)=$

$$=\check{R}_{j,j+1}(t_{j,j+1})\,\check{R}_{\overline{j+1},\bar{j}}(t_{j+1,j})\;h_n(\cdots,t_j,t_{j+1},\cdots)$$

$$(1\le j\le n-1)$$

(2)

$$h_n(\cdots,t_j-\eta,\cdots)=A_n^{(j)}(t_1,\cdots,t_n)h_n(\cdots,t_j,\cdots)$$

(3)

$$\mathcal{P}^{-}_{1,\bar{1}}\cdot h_n(t_1,\cdots,t_n)_{1,\dots,n,\bar{n},\dots,\bar{1}}=s_{1\bar{1}}\;h_{n-1}(t_2,\cdots,t_n)_{2,\dots,n,\bar{n},\dots,\bar{2}}$$

Correlation functions and the reduced qKZ

Here:

$$\check{R}(t) := P\,R(t),$$

$$t_{i,j} := t_i - t_j,$$

$$\mathcal{P}^{\pm} := \tfrac{1}{2}(1 \pm P),$$

$$s := \tfrac{1}{2}(v_+ \otimes v_- - v_- \otimes v_+) \in V \otimes V$$

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and

$$A_n^{(j)}(t_1, \cdots, t_n) :=$$
$$(-1)^n R_{j,j-1}(t_{j,j-1} - \eta) \cdots R_{j,1}(t_{j,1} - \eta) R_{\bar{j},\overline{j+1}}(t_{j,j+1} - \eta) \cdots R_{\bar{j},\bar{n}}(t_{j,n} - \eta)$$
$$\times P_{j,\bar{j}} R_{j,n}(t_{j,n}) \cdots R_{j,j+1}(t_{j,j+1}) R_{\bar{j},\bar{1}}(t_{j,1}) \cdots R_{\bar{j},\overline{j-1}}(t_{j,j-1})\,.$$

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THEOREM

$$h_n(t_1, \cdots, t_n) = e^{\Omega_n(t_1,\cdots,t_n)}\, \mathbf{s}_n$$

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THEOREM

$$h_n(t_1,\cdots,t_n) = e^{\Omega_n(t_1,\cdots,t_n)}\,\mathbf{s}_n$$

where the operator

$$\Omega_n(t_1,\cdots,t_n) := \sum_{i<j} \Omega_n^{(i,j)}(t_1,\cdots,t_n) \quad \text{and} \quad \mathbf{s}_n := \prod_{j=1}^{n} s_{j\bar{j}}$$

Correlation functions and the reduced qKZ

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- **Commutativity:** For distinct indices i_1, j_1, i_2, j_2,

$$\left[\Omega_n^{(i_1,j_1)}(t_1,\cdots,t_n),\Omega_n^{(i_2,j_2)}(t_1,\cdots,t_n)\right]=0$$

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- **Nilpotency:** $\boldsymbol{\Omega}_n^{(i,j)}(t_1,\cdots,t_n)\boldsymbol{\Omega}_n^{(k,l)}(t_1,\cdots,t_n) = 0$

  if $\{i,j\}\cap\{k,l\} \neq \emptyset$

Correlation functions and the reduced qKZ

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LEMMA 1

$$(\Omega_n)^m = 0 \quad \text{for} \quad m > \left[\frac{n}{2}\right]$$

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Lemma 1 means that the expansion in the exponent e^{Ω_n} terminates and only first $\left[\frac{n}{2}\right]$ terms survive.

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REMARK

Lemma 1 means that the expansion in the exponent e^{Ω_n} terminates and only first $\left[\frac{n}{2}\right]$ terms survive.

LEMMA 2

The operator $\Omega_n(t_1, \cdots, t_n)$ is regular at $t_k = t_j$.

Correlation functions and the reduced qKZ

Now let us describe the operators $\Omega_n^{(i,j)}(t_1,\cdots,t_n)$. In order to define them we need several ingredients.

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The operators $\Omega_n^{(i,j)}(t_1, \cdots, t_n)$ consist of "physical part" and "algebraic part"

$$\Omega_n^{(i,j)}(t_1, \cdots, t_n) = \sum_{a=1}^{3} \omega_a(t_{ij})\, X_{a,n}^{(i,j)}(t_1, \cdots, t_n),$$

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- "physical part" is completely fixed by the partition function of the model and described by three functions $\omega_a(t)$

Correlation functions and the reduced qKZ

$$\omega_1(t) := \frac{\partial}{\partial t}\log\varphi(t),\ \omega_2(t) := \frac{\partial}{\partial \eta}\log\varphi(t),\ \omega_3(t) := \frac{\partial}{\partial \tau}\log\varphi(t),$$

Correlation functions and the reduced qKZ

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where we have set $\varphi(t) := \rho(t)^4 \cdot \frac{\theta_1(2\eta-2t)}{\theta_1(2\eta+2t)}$.

Correlation functions and the reduced qKZ

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where we have set $\quad \varphi(t) := \rho(t)^4 \cdot \frac{\theta_1(2\eta - 2t)}{\theta_1(2\eta + 2t)}$.

They are a meromorphic solution of the system of difference equations

$$\omega_1(t-\eta) + \omega_1(t) = q_1(t),$$
$$\omega_2(t-\eta) + \omega_2(t) - \omega_1(t-\eta) = q_2(t),$$
$$\omega_3(t-\eta) + \omega_3(t) = q_3(t),$$

Correlation functions and the reduced qKZ

where

$$q_1(t) := \frac{\partial}{\partial t} \log \psi(t), \; q_2(t) := \frac{\partial}{\partial \eta} \log \psi(t), \; q_3(t) := \frac{\partial}{\partial \tau} \log \psi(t),$$

Correlation functions and the reduced qKZ

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and

$$\psi(t) := \frac{\theta_1(2t)^3\theta_1(2t-4\eta)}{\theta_1(2t-2\eta)^3\theta_1(2t+2\eta)}.$$

Correlation functions and the reduced qKZ

- "algebraic part" is described by three sets of operators $X_{a,n}^{(i,j)}(t_1,\cdots,t_n), \quad a=1,2,3$

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A key role for their description will be played by a family of certain 'transfer matrices' $X_n^{(i,j)}(t_1,\cdots,t_n)$

- "algebraic part" is described by three sets of operators $X^{(i,j)}_{a,n}(t_1,\cdots,t_n),\quad a=1,2,3$

A key role for their description will be played by a family of certain 'transfer matrices' $X^{(i,j)}_{n}(t_1,\cdots,t_n)$

First we define: $$X_n(t_1,\cdots,t_n):=\frac{1}{[t_{1,2}]\prod_{p=3}^{n}[t_{1,p}][t_{2,p}]}\cdot$$
$$\mathrm{Tr}_{\left(\frac{t_{1,2}}{\eta}\right)}\left(T^{[1]}_n\left(\frac{t_1+t_2}{2};t_1,\cdots,t_n\right)\right)P_{12}\mathcal{P}^{-}_{\bar{1}\bar{1}}\mathcal{P}^{-}_{\bar{2}\bar{2}}$$

whose auxiliary spaces have 'fractional dimension'.

Correlation functions and the reduced qKZ

Here we have used 'monodromy matrix'

$$T_n^{[1]}(t;t_1,\ldots,t_n) :=$$

$$L_{\bar{2}}(t-t_2-\eta)\cdots L_{\bar{n}}(t-t_n-\eta)L_n(t-t_n)\cdots L_2(t-t_2)$$

which is the product of L-operators over the Sklyanin algebra.

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which is the product of L-operators over the Sklyanin algebra.

For $i<j$, we define

$$X_n^{(i,j)}(t_1,\cdots,t_n) = X_n^{(j,i)}(t_1,\cdots,t_n) :=$$
$$:= \mathbb{R}_n^{(i,j)}(t_1,\cdots,t_n) X_n(t_i,t_j,t_1,\cdots,\widehat{t_i},\cdots,\widehat{t_j},\cdots,t_n)\cdot$$
$$\mathbb{R}_n^{(i,j)}(t_1,\cdots,t_n)^{-1}.$$

$$\mathbb{R}^{(i,j)}(t_1,\cdots,t_n) :=$$

$$R_{i,i-1}(t_{i,i-1})\cdots R_{i,1}(t_{i,1})R_{j,j-1}(t_{j,j-1})\cdots \widehat{R_{j,i}}\cdots R_{j,1}(t_{j,1})$$

$$\times R_{\overline{i-1},\bar{i}}(t_{i-1,i})\cdots R_{\bar{1}\bar{i}}(t_{1,i})R_{\overline{j-1},\bar{j}}(t_{j-1,j})\cdots \widehat{R_{\bar{i},\bar{j}}}\cdots R_{\bar{1},\bar{j}}(t_{1,j}).$$

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The above mentioned operators $X^{(i,j)}_{a,n}(t_1,\cdots,t_n)$ look

$$c\,X^{(i,j)}_{1,n}(t_1,\cdots,t_n) := X^{(i,j)}_n(t_1,\cdots,t_n) - t_{ij}\Delta^{(i)}_1\,X^{(i,j)}_n(t_1,\cdots,t_n),$$
$$c\,X^{(i,j)}_{2,n}(t_1,\cdots,t_n) = -\eta\Delta^{(i)}_1\,X^{(i,j)}_n(t_1,\cdots,t_n),$$
$$c\,X^{(i,j)}_{3,n}(t_1,\cdots,t_n) := \Delta^{(i)}_\tau\,X^{(i,j)}_n(t_1,\cdots,t_n) - \tau\Delta^{(i)}_1\,X^{(i,j)}_n(t_1,\cdots,t_n),$$

where $c = -2\theta_1'/\theta_1(2\eta)$

Correlation functions and the reduced qKZ

and

$$\Delta_a^{(i)} f(\cdots, t_i, \cdots) = f(\cdots, t_i + a, \cdots) - f(\cdots, t_i, \cdots).$$

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$$\Delta_a^{(i)} f(\cdots, t_i, \cdots) = f(\cdots, t_i + a, \cdots) - f(\cdots, t_i, \cdots).$$

$X_{a,n}^{(ij)}(t_1, \cdots, t_n)$ are doubly periodic in t_k with periods $1, \tau$. The only exception is the case $a = 1$, $k = i$ or j and with respect to the shift by τ, where the transformation law becomes

$$\Delta_\tau^{(i)} X_{1,n}^{(i,j)}(t_1, \cdots, t_n) = \Delta_{-\tau}^{(j)} X_{1,n}^{(i,j)}(t_1, \cdots, t_n) =$$
$$= X_{3,n}^{(i,j)}(t_1, \cdots, t_n).$$

Correlation functions and the reduced qKZ

Conversely we have

$$X_n^{(ij)}(t_1,\cdots,t_n) \;=\; c\left(X_{1,n}^{(i,j)}(t_1,\cdots,t_n)-\frac{t_{ij}}{\eta}\,X_{2,n}^{(i,j)}(t_1,\cdots,t_n)\right).$$

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The separation of $X_n^{(i,j)}$ into two parts $X_{1,n}^{(i,j)}$ and $X_{2,n}^{(i,j)}$ is connected with the structure of the trace function that we still need to define.

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The separation of $X_n^{(i,j)}$ into two parts $X_{1,n}^{(i,j)}$ and $X_{2,n}^{(i,j)}$ is connected with the structure of the trace function that we still need to define.

But first we shall discuss the L-operators.

L-operators and Sklyanin algebra

Sklyanin, (82-83) suggested the following L-operators which satisfy the '$RLL = LLR$'-equation

$$L(t) := \frac{1}{2}\sum_{\alpha=0}^{3} \frac{\theta_{\alpha+1}(2t+\eta)}{\theta_{\alpha+1}(\eta)} S_\alpha \otimes \sigma^\alpha \quad \in \mathcal{A} \otimes \mathrm{End}(V)$$

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where S_α are four generators which satisfy the Sklyanin algebra $\mathcal{A}$

$$[S_0, S_a] = iJ_{bc}(S_b S_c + S_c S_b),$$
$$[S_b, S_c] = i(S_0 S_a + S_a S_0),$$

where (a, b, c) run over cyclic permutations of $(1, 2, 3)$.

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$$J_{bc} = -\frac{J_b - J_c}{J_a} = \varepsilon_a \frac{\theta_1(\eta)^2 \theta_{a+1}(\eta)^2}{\theta_{b+1}(\eta)^2 \theta_{c+1}(\eta)^2},$$

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where

$$\varepsilon_2 = -1, \quad \varepsilon_\alpha = 1 \ (\alpha \neq 2).$$

$\theta_a = \theta_a(0)$ and $\theta_1' = \theta_1'(0)$.

L-operators and Sklyanin algebra

Since the defining relations are homogeneous, $\mathcal{A}$ is a $\mathbb{Z}_{\geq 0}$-graded algebra, $\mathcal{A} = \oplus_{n \geq 0} \mathcal{A}_n$, where the generators S_a all belong to $\mathcal{A}_1$.

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We have also a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading, $\mathcal{A} = \oplus_{(m,n) \in \mathbb{Z}_2 \times \mathbb{Z}_2} \mathcal{A}^{(m,n)}$, defined by the assignment $S_\alpha \in \mathcal{A}^{\bar{\alpha}}$, where

$$\bar{0} = (0,0),\ \bar{1} = (1,0),\ \bar{2} = (1,1),\ \bar{3} = (0,1) \quad \in \mathbb{Z}_2 \times \mathbb{Z}_2.$$

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To make distinction, we call the $\mathbb{Z}_{\geq 0}$-grading and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading ‘degree’ and ‘color’, respectively. Thus S_α has degree 1 and color $\bar{\alpha}$.

L-operators and Sklyanin algebra

There are two Casimir operators of degree 2 and color 0:

$$K_0 = \sum_{\alpha=0}^{3} S_\alpha^2 = 4\frac{\theta_1(\lambda\eta)^2}{\theta_1(2\eta)^2},$$

$$K_2 = \sum_{a=1}^{3} J_a S_a^2 = 4\frac{\theta_1(\lambda\eta+\eta)\theta_1(\lambda\eta-\eta)}{\theta_1(2\eta)^2}$$

λ is the dimension of irreducible representation.

Trace function

We need to consider an analytic continuation of usual trace $\mathrm{tr}_{\mathcal{V}^{(k)}} \pi^{(k)}(A)$ of an element $A \in \mathcal{A}$ as a function of the dimension $k+1$. The precise meaning is as follows.

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- $\mathrm{Tr}_\lambda A\big|_{\lambda=k+1} = \mathrm{tr}_{V(k)}\pi^{(k)}(A)$ holds for all $k \in \mathbb{Z}_{\geq 0}$,
- If $A \in \mathcal{A}_n$, $\mathrm{Tr}_\lambda A$ has the functional form

$$\mathrm{Tr}_{\frac{t}{\eta}} A = \theta_1(t)^n \times \begin{cases} g_{A,0}(t) & (\ n\text{: odd }), \\ g_{A,1}(t) - \frac{t}{\eta} g_{A,2}(t) & (\ n\text{: even }), \end{cases}$$

Trace function

where $g_{A,0}(t)$, $g_{A,2}(t)$ and $g_{A,3}(t) := g_{A,1}(t+\tau) - g_{A,1}(t)$ are elliptic functions with periods $1, \tau$. In addition, $g_{A,1}(t+1) = g_{A,1}(t)$.

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Trace function

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We also set

$$F_{\alpha 3}(t) = \varepsilon_\alpha \theta_{\alpha+1}(\eta)^2 \theta_{\alpha+1}(t+\eta)\theta_{\alpha+1}(t-\eta).$$

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We have also compared our formula for Tr_λ with the results obtained by Fabricius and McCoy, (2004) for $\lambda = 3, 4, 5$.

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- $\mathrm{Tr}_\lambda(AB) = \mathrm{Tr}_\lambda(BA)$,
- $\mathrm{Tr}_\lambda(K_i A) = K_i(\lambda)\mathrm{Tr}_\lambda(A) \quad (i = 0, 2)$,
- $\mathrm{Tr}_\lambda A = 0 \qquad (A \in \mathcal{A}^{(m,n)}, (m,n) \neq (0,0))$.

Trace function

The above formulas are enough to define the trace function of any element of the Sklyanin algebra because the trace function of any "white" combination of the Sklyanin generators may be reduced to the trace function of four simple elements : $1,\ S_0,\ S_0^2,\ S_3^2$
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In the classical limit ($\eta \to 0, \quad \lambda \to \infty, \quad \eta\lambda \equiv t \quad \leftarrow$ finite) , the Sklyanin algebra becomes the algebra of regular functions on an algebraic surface in $\mathbb{C}^4$.

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In the classical limit ($\eta \to 0, \quad \lambda \to \infty, \quad \eta\lambda \equiv t \quad \leftarrow$ finite) , the Sklyanin algebra becomes the algebra of regular functions on an algebraic surface in $\mathbb{C}^4$.

The reduction of the trace is closely connected with the de Rham cohomology of this surface.

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COMPARISON

Using the above results we have got explicit formulae for $n = 2$ and $n = 3$ and compared numerically with the known results obtained by Lashkevich and Pugai using free field representation of the vertex operators
Lashkevich and Pugai, Nucl. Phys. (1998)

REMARK 1

The operators $\Omega_n^{(i,j)}$ satisfy sort of fermionic algebra which via Jordan-Wigner transformation may lead to a fermionic representation . The question which arises here is: if there exists some connection with fermions then what is the meaning of these fermions.

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REMARK 2

Since the functions $h_n(t_1, \cdots, t_n)$ are related to the density matrix of quantum mechanical problem for the spin chain with two subsystems the operator Ω_n resembles Hamilton operator for this problem.

Outlook

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- Second, it is interesting to investigate SOS counterpart. We anticipate essential simplification here.
- The third important question is how to generalize all above results to the 'root-of-unity' case.

Outlook

- generalization of the above scheme to the case of finite temperature using recent results by Göhmann, Klümper and Seel: hep-th/0405089, hep-th/0406611

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- generalization to the case of dynamic correlations. One can mention recent results by Kitanine, Maillet, Slavnov, Terras: hep-th/0407108, hep-th/0407223, hep-th/0406190